Chapter 3

Harmonic Analysis

⟨ch:Harmonic_Analysis⟩

Recall that if $f \in L^2([0\,,2\pi]^n)$, then we can write f as

$$f(x) = (2\pi)^{-n} \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{-ik\cdot x} \hat{f}_k, \tag{3.1}$$ F:L

where $\hat{f}_k := \int_{[0,2\pi]^n} \mathrm{e}^{ik\cdot x} f(x)\,\mathrm{d}x$ denotes the "kth Fourier coefficient" of f, and convergence holds in $L^2([0\,,2\pi]^n)$; that is, $\int_{[0,2\pi]^n} |f(x) - f_N(x)|^2\mathrm{d}x \to 0$ as $N \to \infty$, where $f_N(x) := \sum_{\|k\|\leqslant N} \exp\{ik\cdot x\}\hat{f}_k$.

Eq. (3.1) is one of the many possible starting points of the theory of harmonic analysis in the Lebesgue space $[0\,,2\pi]^n$. In this chapter we develop a parallel theory for the Gauss space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\,, \mathrm{P}_n)$. Problems 15 through 20 work out the analogous details for "Poisson" spaces. And other distributional spaces are also possible; see XXX for more discussion on this topic.

1 Hermite Polynomials in Dimension One

Before we discuss the general n-dimensional case, let us consider the special case that $n = 1$. We may observe the following elementary computations:

$$\gamma_1'(x) = -x\gamma_1(x), \quad \gamma_1''(x) = (x^2-1)\gamma_1(x), \quad \gamma_1'''(x) = -(x^3-3x)\gamma_1(x), \quad \text{etc.}$$

It follows from these computations, and from induction, that the kth derivative of γ_1 satisfies

$$\gamma_1^{(k)}(x) = (-1)^k H_k(x)\gamma_1(x) \qquad [k \geqslant 0,\, x\in\mathbb{R}], \tag{3.2}$$ def:Hermite

where H_k is a polynomial of degree at most k. Moreover,

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad \text{etc.}$$

Definition 1.1. H_k is called the *Hermite polynomial* of degree $k \geqslant 0$.

Be warned that some authors normalize their Hermite polynomials differently than has been done here. Therefore, our notation, and normalizing constants, might differ from theirs in certain places.

The following basic lemma records some of the salient features of Hermite polynomials.

⟨lem:Hermite⟩ **Lemma 1.2.** *For all $x \in \mathbb{R}$ and $k \in \mathbb{Z}_+$:*

⟨lem:Hermite:1⟩ *1.* $H_{k+1}(x) = xH_k(x) - H_k'(x)$;

⟨lem:Hermite:2⟩ *2.* $H_{k+1}'(x) = (k+1)H_k(x)$; *and*

⟨lem:Hermite:3⟩ *3.* $H_k(-x) = (-1)^k H_k(x)$.

This simple lemma teaches us a great deal about Hermite polynomials. For instance, we learn from part 1 and induction that

$$H_k \text{ is a polynomial of exact degree } k \text{ for every } k \geqslant 0,$$

and the following *Rodriguez formula* holds for all $k \geqslant 0$ and $x \in \mathbb{R}$:

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x). \tag{3.3}$$ Rodriguez

Finally, every H_k is *monic*; that is, the coefficient of x^k in $H_k(x)$ is one for all $k \geqslant 0$. Other properties of Hermite polynomials will unfold themselves in due time. For the time being, let us prove Lemma 1.2.

Proof. We prove part 1 of the lemma by direct computation:

$$\begin{aligned}
(-1)^{k+1} H_{k+1}(x)\gamma_1(x) &= \gamma_1^{(k+1)}(x) && \text{[by (3.2)]}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\gamma_1^{(k)}(x)\\
&= (-1)^k \frac{\mathrm{d}}{\mathrm{d}x}\left[H_k(x)\gamma_1(x)\right] && \text{[by (3.2)]}\\
&= (-1)^k \left[H_k'(x)\gamma_1(x) + H_k(x)\gamma_1'(x)\right]\\
&= (-1)^k \left[H_k'(x) - xH_k(x)\right]\gamma_1(x),
\end{aligned}$$

where the last line follows from a third appeal to (3.2), together with the fact that $H_1(x) = x$. Divide both sides by $(-1)^{k+1}\gamma_1(x)$ to complete the proof of part 1.

Part 2 is clearly correct when $k = 0$. We now apply induction: Suppose $H_{j+1}'(x) = (j+1)H_j(x)$ for all $0 \leqslant j \leqslant k$. We plan to prove this for $j = k+1$. By part 1 and the induction hypothesis, the Rodriguez formula (3.3) holds. Therefore, we can differentiate the latter formula in order to find that

$$\begin{aligned}
H_{k+1}'(x) &= H_k(x) + xH_k'(x) - kH_{k-1}'(x)\\
&= H_k(x) + kxH_{k-1}(x) - kH_{k-1}'(x),
\end{aligned}$$

thanks to a second appeal to the induction hypothesis. Because of Part 1, $xH_{k-1}(x) - H_{k-1}'(x) = H_k(x)$. This proves that $H_{k+1}'(x) = (k+1)H_k(x)$, and part 2 follows.

We apply parts 1 and 2 of the lemma, and induction, in order to see that H_k is odd [and H_k' is even] if and only if k is. This proves part 3. $\square$

The following is the *raison d'être* for our study of Hermite polynomials. Specifically, it states that the sequence $\{H_k\}_{k=0}^\infty$ plays the same sort of harmonic-analyatic role in the 1-dimensional Gauss space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathrm{P}_1)$ as do the complex exponentials in Lebesgue spaces.

⟨th:Hermite:1⟩ **Theorem 1.3.** *The normalized Hermite polynomials* $\left\{H_k/\sqrt{k!}\right\}_{k=0}^\infty$ *form a complete, orthonormal basis for* $L^2(\mathrm{P}_1)$.

Before we prove Theorem 1.3 let us mention the following corollary.

⟨co:Hermite:1⟩ **Corollary 1.4.** *For every* $f \in L^2(\mathrm{P}_1)$,

$$f = f(Z) = \sum_{k=0}^{\infty} \frac{1}{k!} \langle f \,, H_k \rangle_{L^2(\mathrm{P}_1)} H_k(Z) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{E}\left[f H_k \right] H_k \qquad a.s.$$

To prove this we merely apply Theorem 1.3 and the Riesz–Fischer theorem. Next is another corollary which also has a probabilistic flavor.

⟨co:Hermite:Wiener:1⟩ **Corollary 1.5** (Wiener XXX)**.** *For all* $f, g \in L^2(\mathrm{P}_1)$,

$$\mathrm{E}[fg] = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{E}[fH_k]\, \mathrm{E}[gH_k] \quad \textit{and} \quad \mathrm{Cov}(f\,, g) = \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{E}[fH_k]\, \mathrm{E}[gH_k].$$

Proof. Multiply both sides of the first identity of Corollary 1.4 by $g(x)$ and integrate $[\mathrm{dP}_1]$ in order to obtain the identity,

$$\langle g \,, f \rangle_{L^2(\mathrm{P}_1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \langle f \,, H_k \rangle_{L^2(\mathrm{P}_1)} \langle g \,, H_k \rangle_{L^2(\mathrm{P}_1)}.$$

The exchange of sums and integrals is justified by Fubini's theorem. The preceding is another way to say the first result. The second follows from the first and the fact that $H_0 \equiv 1$. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. Recall the adjoint operator A from (2.4) on page 36. Presently, $n = 1$; therefore, in this case, A maps a scalar function to a scalar function. Since polynomials are in the domain of the definition of A [Chapter 2, Proposition 3.3], parts 1 and 2 of Lemma 1.2 respectively say that:[1]

$$H_{k+1} = AH_k \quad \text{and} \quad DH_{k+1} = (k+1)H_k \qquad \text{for all } k \geqslant 0. \tag{3.4}$$ A:D:H

Consequently,

$$\mathrm{E}(H_k^2) = \mathrm{E}\left[H_k \cdot A(H_{k-1}) \right] = \mathrm{E}\left[D(H_k) \cdot H_{k-1} \right] = k\, \mathrm{E}\left[H_{k-1}^2 \right].$$

Since $\mathrm{E}(H_0^2) = 1$, induction shows that $\mathrm{E}(H_k^2) = k!$ for all integers $k \geqslant 0$.

Next we prove that

$$\mathrm{E}(H_k H_{k+\ell}) = 0 \qquad \text{for integers } \ell > 0, k \geqslant 0.$$

By (3.4),

$$\mathrm{E}(H_k H_{k+\ell}) = \mathrm{E}\left[H_k \cdot A(H_{k+\ell-1}) \right] = \mathrm{E}\left[D(H_k) H_{k+\ell-1} \right] = k\, \mathrm{E}\left[H_{k-1} H_{k+\ell-1} \right].$$

[1] It is good to remember that H_k plays the same role in the Gauss space $(\mathbb{R}\,, \mathcal{B}(\mathbb{R})\,, \mathrm{P}_1)$ as does the monomial x^k in the Lebesgue space. Therefore, $DH_{k+1} = (k+1)H_k$ is the Gaussian analogue of the statement that $\mathrm{d}(x^{k+1})/\mathrm{d}x = (k+1)x^k$. As it turns out the adjoint operator behaves a little like an integral operator, and the identity $AH_k = H_{k+1}$ is the Gaussian analogue of the anti-derivative identity $\int x^k \,\mathrm{d}x \propto x^{k+1}$, valid in Lebesgue space.

Now iterate this identity to find that

$$\mathrm{E}\left(H_k H_{k+\ell}\right) = k!\,\mathrm{E}\left[H_0 H_\ell\right] = k!\int_{-\infty}^{\infty} H_\ell(x)\gamma_1(x)\,\mathrm{d}x = 0,$$

since $H_\ell\,\gamma_1 = (-1)^\ell\gamma_1^{(\ell)}$, thanks to (3.2). It follows that $\left\{H_k/\sqrt{k!}\right\}_{k=0}^{\infty}$ is an orthonormal sequence of elements of $L^2(\mathrm{P}_1)$.

In order to complete the proof, we need to show the orthonormal basis is complete. We do this in a standard way. Namely, we suppose that $f \in L^2(\mathrm{P}_1)$ is orthogonal in $L^2(\mathrm{P}_1)$ to H_k for all $k \geqslant 0$, and then proceed to prove that, as a consequence, $f = 0$ almost surely $[\mathrm{P}_1]$.

Part 1 of Lemma 1.2 shows that $H_k(x) = x^k - p(x)$ where p is a polynomial of degree $k-1$ for every $k \geqslant 1$. Consequently, the span of $H_0, \ldots, H_k$ is the same as the span of the monomials $1, x, \cdots, x^k$ for all $k \geqslant 0$. In particular, $\int_{-\infty}^{\infty} f(x)x^k\gamma_1(x)\,\mathrm{d}x = 0$ for all $k \geqslant 0$. Multiply both sides by $(-it)^k/k!$ and add over all $k \geqslant 0$ in order to see that

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-itx}\gamma_1(x)\,\mathrm{d}x = 0 \qquad \text{for all } t \in \mathbb{R}. \tag{3.5}$$ pre:Hermite

If the Fourier transform $\hat{g}$ of a function $g \in C_c(\mathbb{R})$ is absolutely integrable, then by the inversion theorem of Fourier transforms,

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-itx}\hat{g}(t)\,\mathrm{d}t \qquad \text{for all } x \in \mathbb{R}.$$

Multiply both sides of (3.5) by $\hat{g}(t)$ and integrate $[\mathrm{d}t]$ in order to see from Fubini's theorem that $\int fg\,\mathrm{dP}_1 = 0$ for all $g \in C_c(\mathbb{R})$ such that $\hat{g} \in L^1(\mathbb{R})$. Since the class of such functions g is dense in $L^2(\mathrm{P}_1)$, it follows that $\int fg\,\mathrm{dP}_1 = 0$ for every $g \in L^2(\mathrm{P}_1)$. Set $g \equiv f$ to see that $f = 0$ a.s. □

Finally, let us mention one more important corollary.

⟨co:Nash⟩ **Corollary 1.6** (A Poincaré Inequality)**.** *For all* $f \in \mathbb{D}^{1,2}(\mathrm{P}_1)$,

$$\mathrm{Var}(f) \leqslant \mathrm{E}\left(|Df|^2\right).$$

Proof. By Corollary 1.5 and (3.4),

$$\begin{aligned}\mathrm{Var}(f) &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\,\left|\mathrm{E}[fH_{k+1}]\right|^2 = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\,\left|\mathrm{E}[fA(H_k)]\right|^2\\ &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\,\left|\mathrm{E}[D(f)H_k]\right|^2 \leqslant \sum_{k=0}^{\infty}\frac{1}{k!}\,\left|\mathrm{E}[D(f)H_k]\right|^2.\end{aligned}$$

The right-most quantity is equal to $\mathrm{E}(|Df|^2)$, thanks to Corollary 1.5. □

2 Hermite Polynomials in General Dimensions

For every $k \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$ define

$$\mathcal{H}_k(x) := \prod_{j=1}^{n} H_{k_j}(x_j) \qquad [x \in \mathbb{R}^n].$$

These are n-variable extensions of Hermite polynomials. Though, when $n = 1$, we will continue to write $H_k(x)$ in place of $\mathcal{H}_k(x)$ in order to distinguish the multi-dimensional case from the case $n = 1$.

Clearly, $x \mapsto \mathcal{H}_k(x)$ is a polynomial, in n variables, of degree k_j in the variable x_j. For instance, when $n = 2$,

$$\mathcal{H}_{(0,0)}(x) = 1, \qquad \mathcal{H}_{(1,0)}(x) = x_1, \qquad \mathcal{H}_{(0,1)}(x) = x_2, \tag{3.6) \{?\}$$

$$\mathcal{H}_{(1,1)}(x) = x_1x_2, \qquad \mathcal{H}_{(1,2)}(x) = x_1(x_2^2 - 1), \ldots . \tag{3.7) \{?\}$$

Because each measure P_n has the product form $\mathrm{P}_n = \mathrm{P}_1 \times \cdots \times \mathrm{P}_1$, Theorem 1.3 immediately extends to the following.

⟨th:Hermite⟩ **Theorem 2.1.** *Define $k! := \prod_{q=1}^n k_q!$ for all $k \in \mathbb{Z}_+^n$. Then, for every integer $n \geqslant 1$, the collection $\left\{\mathcal{H}_k/\sqrt{k!}\right\}_{k\in\mathbb{Z}_+^n}$ is a complete, orthonormal basis in $L^2(\mathrm{P}_n)$.*

Corollary 1.4 has the following immediate extension.

⟨co:Hermite⟩ **Corollary 2.2.** *For every $n \geqslant 1$ and $f \in L^2(\mathrm{P}_n)$,*

$$f = \sum_{k\in\mathbb{Z}_+^n} \frac{\mathrm{E}(f\mathcal{H}_k)}{k!}\, \mathcal{H}_k \quad \textit{almost surely,}$$

where the infinite sum converges in $L^2(\mathrm{P}_n)$.

Similarly, the following immediate extension of Corollary 1.5 computes the covariance between two arbitrary square-integrable random variables in the Gauss space.

⟨co:Hermite:Wiener⟩ **Corollary 2.3** (Wiener XXX)**.** *For all $n \geqslant 1$ and $f, g \in L^2(\mathrm{P}_n)$,*

$$\mathrm{E}[fg] = \sum_{k\in\mathbb{Z}_+^n} \frac{1}{k!}\, \mathrm{E}[f\mathcal{H}_k]\, \mathrm{E}[g\mathcal{H}_k] \quad \textit{and} \quad \mathrm{Cov}(f\,, g) = \sum_{\substack{k\in\mathbb{Z}_+^n\\ k\neq 0}} \frac{1}{k!}\, \mathrm{E}(f\mathcal{H}_k)\, \mathrm{E}(g\mathcal{H}_k).$$

And the following generalizes Corollary 1.6 to several dimensions.

⟨pr:Nash⟩ **Proposition 2.4** (The Poincaré Inequality)**.** *For all $f \in \mathbb{D}^{1,2}(\mathrm{P}_n)$,*

$$\mathrm{Var}(f) \leqslant \mathrm{E}\left(\|Df\|^2\right).$$

Proof. By Corollary 2.2, the following holds a.s. for all $1 \leqslant q \leqslant n$:

$$D_qf = \sum_{k\in\mathbb{Z}_+^n} \frac{\mathrm{E}[D_q(f)\mathcal{H}_k]}{k!}\, \mathcal{H}_k = \sum_{k\in\mathbb{Z}_+^n} \frac{\mathrm{E}[fA_q(\mathcal{H}_k)]}{k!}\, \mathcal{H}_k,$$

where we recall A_q denotes the qth coordinate of the vector-valued adjoint operator. By orthogonality and (3.4),

$$\begin{aligned}
\mathrm{E}\left(\|Df\|^2\right) &= \sum_{q=1}^n \sum_{k\in\mathbb{Z}_+^n} \frac{1}{k!}\, |\mathrm{E}[fA_q(\mathcal{H}_k)]|^2 \\
&= \sum_{q=1}^n \sum_{k\in\mathbb{Z}_+^n} \frac{1}{k!} \left| \mathrm{E}\left[f(Z)H_{k_q+1}(Z_q) \prod_{\substack{1\leqslant \ell\leqslant n\\ \ell\neq q}} H_{k_\ell}(Z_\ell) \right] \right|^2 .
\end{aligned}$$

Fix an integer $1 \leqslant q \leqslant n$ and relabel the inside sum as $j_\ell := k_\ell$ if $\ell \neq q$ and $j_q := k_q + 1$. In this way we find that

$$\begin{aligned}
\mathrm{E}\left(\|Df\|^2\right) &\geqslant \sum_{q=1}^n \sum_{\substack{j \in \mathbb{Z}_+^n \\ j_q \geqslant 1}} \frac{1}{j_1! \cdots j_n!} \left| \mathrm{E}\left[f(Z) H_{j_q}(Z_q) \prod_{\substack{1 \leqslant \ell \leqslant n \\ \ell \neq q}} H_{j_\ell}(Z_\ell) \right] \right|^2 \\
&= \sum_{q=1}^n \sum_{\substack{j \in \mathbb{Z}_+^n \\ j_q \geqslant 1}} \frac{1}{j_1! \cdots j_n!} \left| \mathrm{E}\left[f \mathcal{H}_j \right] \right|^2 .
\end{aligned}$$

using only the fact that $1/(j_q - 1)! > 1/j_q!$. This completes the proof since the right-hand side is simply

$$\sum_{j \in \mathbb{Z}_+^n} \frac{1}{j_1! \cdots j_n!} \left| \mathrm{E}\left[f \mathcal{H}_j \right] \right|^2 - |\mathrm{E}[f \mathcal{H}_0]|^2 ,$$

which is equal to the variance of $f(Z)$ [Corollary 2.3]. □

Consider a Lipschitz-continuous function $f : \mathbb{R}^n \to \mathbb{R}$. Recall [Example 1.6, page 32] that this means that $\mathrm{Lip}(f) < \infty$, where

$$\mathrm{Lip}(f) := \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|}.$$

Since $f \in \mathbb{D}^{1,2}(\mathrm{P}_n)$ and $\|Df\| \leqslant \mathrm{Lip}(f)$ a.s., the Poincaré inequality has the following ready consequence.

⟨co:Nash:Lip⟩ **Corollary 2.5.** *For every Lipschitz-continuous function $f : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathrm{Var}(f) \leqslant |\mathrm{Lip}(f)|^2 .$$

If f is almost constant, then $f \approx \mathrm{E}(f)$ with high probability and hence $\mathrm{Var}(f) \approx 0$. The preceding estimate is an *a priori* way of saying that "in high dimensions, most Lipschitz-continuous functions are almost constant." This assertion is substantiated further by the following two examples.

Example 2.6. The function $f(x) := n^{-1} \sum_{i=1}^n x_i$ is Lipschitz continuous and $\mathrm{Lip}(f) = 1/\sqrt{n}$. In this case, Corollary 2.5 implies that

$$\mathrm{Var}\left(n^{-1} \textstyle\sum_{i=1}^n Z_i\right) \leqslant n^{-1},$$

which is in fact an identity. This example shows that the bound in the Poincaré inequality can be attained. The fact that f is "almost constant" is another way to state Khintchine's weak law of large numbers.

Example 2.7. For a more interesting example consider either the function $f(x) := \max_{1 \leqslant i \leqslant n} |x_i|$ or the function $g(x) := \max_{1 \leqslant i \leqslant n} x_i$. Both f and g are Lipschitz-continuous functions with Lipschitz constant at most 1. The Poincaré inequality implies that $\mathrm{Var}(M_n) \leqslant 1$,[2] where M_n denotes either $\max_{1 \leqslant i \leqslant n} Z_i$ or $\max_{1 \leqslant i \leqslant n} |Z_i|$.

[2]This bound is sub optimal. The optimal bound is $\mathrm{Var}(M_n) = O(1/\log n)$. For more information on this see part (b) of Problem 19 on page 27.

This is a non-trivial result about, for example, the absolute size of the *centered* random variable $M_n - \mathrm{E}\, M_n$. The situation changes completely once we remove the centering. Indeed by Proposition 1.3 (p. 7) and Jensen's inequality,

$$\mathrm{E}(M_n^2) \geqslant |\mathrm{E}(M_n)|^2 \sim 2\log n \qquad \text{as } n\to\infty.$$

Similar examples can be constructed for more general Gaussian random vectors than Z, thanks to the following.

⟨pr:Poincare:X⟩ **Proposition 2.8.** *Let Q be a positive semidefinite matrix, and define λ_* to be its largest eigenvalue. If X is distributed as $\mathrm{N}_n(0\,,Q)$, then*

$$\mathrm{Var}[f(X)] \leqslant \lambda_*\, \mathrm{E}\left(\|(Df)(X)\|^2\right) \qquad \textit{for every } f\in \mathbb{D}^{1,2}(\mathrm{P}_n).$$

Proof. We can write $Q = S^2$ where S is a symmetric $n\times n$ matrix; that is, S is a square root of Q. Define $g(x) := f(Sx)$ for every $x\in\mathbb{R}^n$, and observe that: (i) X has the same distribution as SZ; and therefore (ii) $\mathrm{Var}(f(X)) = \mathrm{Var}(g(Z)) \leqslant \mathrm{E}(\|(Dg)(Z)\|^2)$ thanks to Proposition 2.4. By the chain rule, $(Dg)(Z) = (Df)(SZ)S$, whence

$$\|(Dg)(Z)\|^2 = \langle (Df)(SZ)S\,,(Df)(SZ)S\rangle_{\mathbb{R}^n} = \langle (Df)(SZ)\,,(Df)(SZ)Q\rangle_{\mathbb{R}^n},$$

thanks to the facts that $Q = S^2$ and S is symmetric.[3] Since Q is symmetric, Rayleigh's principle yields $\langle x, xQ\rangle_{\mathbb{R}^n} \leqslant \lambda_*\|x\|^2$ for all $x\in\mathbb{R}^n$. Set $x := (Df)(SZ)$ to see that

$$\mathrm{E}(\|(Dg)(Z)\|^2) \leqslant \lambda_*\,\mathrm{E}(\|(Df)(SZ)\|^2),$$

which is equal to $\lambda_*\,\mathrm{E}(\|(Df)(X)\|^2)$. □

The above proposition is sharp: For example, consider a function f such that Df is constant almost everywhere, and lies in the span of the top eigenvalue of Q. However, the proposition can be sharpened for certain specific choices of f. The next proposition highlights this assertion.

⟨pr:Var:max⟩ **Proposition 2.9.** *If X has a $\mathrm{N}_n(0\,,Q)$ distribution, then*

$$\mathrm{Var}(M_n) \leqslant \max_{1\leqslant i\leqslant n} \mathrm{Var}(X_i),$$

where M_n denotes either $\max_{1\leqslant i\leqslant n} X_i$ or $\max_{1\leqslant i\leqslant n}|X_i|$.

Proof. Let S denote a symmetric square root of Q, and define $f(x) = \max_{1\leqslant i\leqslant n} x_i$. According to the proof of Proposition 2.8,

$$\mathrm{Var}\left(\max_{1\leqslant i\leqslant n} X_i\right) = \mathrm{Var}[f(SZ)] \leqslant \mathrm{E}\left[\langle (Df)(SZ)\,,(Df)(SZ)Q\rangle_{\mathbb{R}^n}\right].$$

We saw in Example 1.8, p. 33 that Df is Lebesgue-almost-everywhere a standard basis vector of $\mathbb{R}^n$, and hence $\langle (Df)(SZ)\,,(Df)(SZ)Q\rangle_{\mathbb{R}^n}$ is P_n-almost surely a diagonal entry of Q. This implies the proposition in the case that $M_n = \max_{1\leqslant i\leqslant n} X_i$. The case that $f(x) = \max_{1\leqslant i\leqslant n}|x_i|$ is handled the same way, except that now Df is ± 1 times some standard basis element of $\mathbb{R}^n$. □

[3]Needless to say, $\langle a\,,b\rangle_{\mathbb{R}^n} := \sum_{i=1}^n a_i b_i$ for all $a,b\in\mathbb{R}^n$.

3 Wick's Formula

Theorem 1.3 shows that the (normalized) Hermite polynomials form a complete, orthonormal basis for $L^2(\mathrm{P}_n)$. From this a straightforward change-of-variables argument allows us to find a similar basis for any $L^2(\mathrm{Q}_n)$, where Q_n is the law of a $\mathrm{N}_n(0,\Gamma)$ random variable with Γ being full rank. Here is the statement of that fact.

⟨co:Hermite:correlated⟩ **Corollary 3.1.** *Let Q_n denote the law of a $\mathrm{N}_n(0,\Gamma)$ random variable with Γ full rank. Define functions $\mathcal{C}$ by $\mathcal{C}_k(x) = \mathcal{H}_k(\Gamma^{-1/2}x)$ for $k \in Z_+^n$. Then $\{\mathcal{C}_k/\sqrt{k!}\}_{k\in\mathbb{Z}_+^n}$ form a complete, orthonormal basis for $L^2(\mathrm{Q}_n)$. Hence for any $f \in L^2(\mathrm{Q}_n)$*

$$f = \sum_{k\in\mathbb{Z}_+^n} \frac{\mathrm{E}[f\mathcal{C}_k]}{k!}\mathcal{C}_k.$$

Corollary 3.1 is of course a simple consequence of Theorem 2.1 and Corollary 2.2. In practice, for a general f it is rarely possible to explicitly compute this decomposition of f since the coefficients $\mathrm{E}[f\mathcal{C}_k]$ involve complicated integrals. When f is a polynomial there is a combinatorial way of organizing a variant of this decomposition that goes by the name of Wick's formula. In fact it is a straightforward extension of the Isserlis formula of Chapter 1, Theorem 5.1, and we have already seen the essential ingredients of Wick's formula in Chapter 1, Lemma 5.2.

Given the polynomials $\mathcal{C}_k$ from above we begin by defining, for integers $j \geqslant 1$, the linear subspaces $\mathcal{P}_j$ of $L^2(\mathrm{Q}_n)$ by

$$\mathcal{P}_j = \mathrm{Span}\{\mathcal{C}_k : |k| = j\},$$

where $|k| = \sum_{m=1}^n k_m$. We set $\mathcal{P}_0$ to be the space of constant functions, and then Corollary 3.1 can be rephrased as saying that $L^2(\mathrm{Q}_n)$ decomposes into the orthogonal sum of the subspaces $\mathcal{P}_j, j \geqslant 0$. Symbolically this is written as

$$L^2(\Omega,\mathcal{F},\mathrm{Q}_n) = \bigoplus_{j=0}^{\infty} \mathcal{P}_j, \tag{3.8}$$ eq:Wiener_chaos_defn

where $\mathcal{F}$ is the σ-algebra generated by $X_1,\dots,X_n$.

A useful alternative description of the spaces $\mathcal{P}_j$ is that if $X = (X_1,\dots,X_n)$ is a collection of mean zero, jointly Gaussian variables, then $\mathcal{P}_j$ is simply the linear subspace of all degree j polynomials in the variables $X_1,\dots,X_n$ that are uncorrelated with all degree $< j$ polynomials in $X_1,\dots,X_n$. This definition has the advantage that it does not require the explicit basis of polynomials $\{\mathcal{C}_k : |k| = j\}$ to define $\mathcal{P}_j$, and this "basis free" definition does not require that the covariance matrix of $X_1,\dots,X_n$ has full rank. In particular it allows for some of the X_i to be equal. Nonetheless the decomposition (3.8) still holds, and simply states that every function of $X_1,\dots,X_n$ with finite variance can be uniquely written as a (possibly) infinite sum of polynomials, one from each $\mathcal{P}_j$. The equality (3.8) is often called the *Wiener chaos decomposition* of the L^2 space.

Wick's formula explains how the decomposition works for polynomial functions of the Gaussian variables. A straightforward implication of (3.8) is that a polynomial of degree ℓ in $X_1,\dots,X_n$ (noting that the degree of a monomial is the sum of its exponents, and the degree of a polynomial is the maximum degree of all monomials appearing in it) can be uniquely decomposed into a sum of elements from $\mathcal{P}_0,\mathcal{P}_1,\dots,\mathcal{P}_\ell$. As noted above the part in $\mathcal{P}_j$ is simply the orthogonal projection of the polynomial onto $\mathcal{P}_j$, in the $L^2(\mathrm{Q}_n)$ inner product. To express the projection we define projection operators

$$\pi_j : L^2(\Omega, \mathcal{F}, Q_n) \to \mathcal{P}_j.$$

Thus for every random variable $Y \in L^2(\Omega, \mathcal{F}, Q_n)$ we can use the projection operator π_j to uniquely write $Y = U + V$, where $U \in \mathcal{P}_j$ and U, V are orthogonal in the $L^2(Q_n)$ inner product. Of course $U = \pi_j Y$ and $V = Y - \pi_j Y$. Recall that projection operators are linear operators, so to explain Wick's formula for polynomials it is enough to consider what the projection operators π do to monomials of the form

$$f(X_1, \ldots, X_n) = X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n}$$

with the a_k being non-negative integers that sum to ℓ. Wick's formula gives both an expression for $\pi_\ell f$ and a way of writing f as a sum of elements from $\mathcal{P}_j, j \leqslant \ell$. As for the Isserlis formula it is a combinatorial expression involving matchings. For the statement and proof of Wick's formula it is useful to introduce notation to describe matchings. We write a partial matching of a set A as $\mathfrak{m} \oplus \mathfrak{u}$, where $\mathfrak{m}$ is the perfect matching of a subset of A and $\mathfrak{u}$ is the remaining unmatched pairs. Lastly, to state the formula it is easier to allow for repetitions among the variables rather than using the powers to express the polynomial. Thus in the following remember that some of the X_i can be the same.

?⟨th:Wick⟩? **Theorem 3.2** (Wick, YYYY)**.** *Let $X_1, \ldots, X_n$ have a $\mathrm{N}_n(0, \Gamma)$ distribution, where Γ is an $n \times n$ covariance matrix. Then*

$$\pi_n(X_1 X_2 \ldots X_n) = \sum_{\mathfrak{m} \oplus \mathfrak{u}} (-1)^{|\mathfrak{m}|} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j} \prod_{k \in \mathfrak{u}} X_k. \tag{3.9}$$ eqn:Wick_projection

where the sum ranges over all partial matchings $\mathfrak{m} \oplus \mathfrak{u}$ of $\{1, \ldots, n\}$, and $|\mathfrak{m}|$ denotes the number of matched pairs in the partial matching. Furthermore, the decomposition of the regular product $X_1 X_2 \ldots X_n$ into orthogonal parts is given by

$$X_1 X_2 \ldots X_n = \sum_{\mathfrak{m} \oplus \mathfrak{u}} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j} \, \pi_{|\mathfrak{u}|} \left(\prod_{k \in \mathfrak{u}} X_k \right), \tag{3.10}$$ eqn:Wick_decomposition

where $\pi_{|\mathfrak{u}|}$ is the projection to the Wiener chaos of the number of unmatched terms $|\mathfrak{u}|$.

Thus the theorem contains two distinct parts. The first is that the projection is itself a polynomial in the variables X_i, while the second is that every polynomial in Gaussian variables term can be written uniquely as a sum of polynomials in the same variables with the property that any two terms in the sum are uncorrelated.

Some examples are in order. Our first example is a "trivial" one.

Example 3.3. A single Gaussian variable X_1 has to be left unmatched, and hence

$$X_1 = \pi_1(X_1),$$

as expected. By linearity this extends to the identity $\pi_1(aX_1 + b) = a\pi_1(X_1)$, valid for all real numbers a and b.

For a more interesting example consider the following.

Example 3.4. If $(X_1\,, X_2)$ has a Gaussian distributions, then X_1 and X_2 can either be matched or unmatched. This observation readily leads us to the formula

$$\pi_2(X_1X_2) = X_1X_2 - \mathrm{E}[X_1X_2].$$

This is equivalent to the assertion that $X_1X_2 = \pi_2(X_1X_2) + \mathrm{E}[X_1X_2]$, which is the orthogonal decomposition of X_1X_2. These formulas continue to hold when $X_1 = X_2 = X$. In this way we are led to the well-known formula:

$$\pi_2(X^2) = X^2 - \mathrm{E}[X^2] = X^2 - \mathrm{Var}(X).$$

In fact it can be checked by that if X is a standard normal random variable and $n \geqslant 1$ an integer then

$$\pi_n(X^n) = H_n(X), \tag{3.11}$$

eq:Hermite_projection_for

for H_n the one-dimensional Hermite polynomial of degree n. To see this in the $n = 3$ case use the formula for X_1, X_2, X_3

$$\pi_3(X_1X_2X_3) = X_1X_2X_3 - \mathrm{E}[X_1X_2]X_3 - \mathrm{E}[X_1X_3]X_2 - \mathrm{E}[X_2X_3]X_1$$

and set $X_1 = X_2 = X_3 = X$. Similarly, for four variables there are either no matchings, exactly one matching, or exactly two matchings, which leads to the expression

$$\begin{aligned}\pi_4(X_1X_2X_3X_4) =& X_1X_2X_3X_3 - \mathrm{E}[X_1X_2]X_3X_4 - \mathrm{E}[X_1X_3]X_2X_4 - \mathrm{E}[X_1X_4]X_2X_3\\ &- \mathrm{E}[X_2X_3]X_1X_4 - \mathrm{E}[X_2X_4]X_1X_3 - \mathrm{E}[X_3X_4]X_1X_2\\ &+ \mathrm{E}[X_1X_2]\,\mathrm{E}[X_3X_4] + \mathrm{E}[X_1X_3]\,\mathrm{E}[X_2X_4] + \mathrm{E}[X_1X_4]\,\mathrm{E}[X_2X_3].\end{aligned}$$

By setting $X_1 = X_2 = X_3 = X_4 = X$ the right hand side becomes exactly $H_4(X)$.

We also observe that equation (3.10) contains the Isserlis formula of Theorem 5.1, Chapter 1. On the right hand side of (3.10) every partial matching that is not a perfect matching necessarily contains a Wick product, and that Wick product is in $\mathcal{P}_j$ for some $j > 0$. Therefore it is orthogonal to $\mathcal{P}_0$ and hence mean zero. Thus after taking expected value of (3.10) only the perfect matching terms remain on the right hand side, which produces precisely the Isserlis formula.

Proof. We prove (3.9) first. Let Y be the right hand side of (3.9). We will first show that it is in $\mathcal{P}_n$. We see that Y is a polynomial of degree n since it contains the term $X_1 \dots X_n$, corresponding to the case when no terms are matched, and since all other terms have at least one pair matched their degree is stricly less than n. Thus we only need to show that Y is orthogonal to all polynomials of degree $< n$. We will start by showing it is orthogonal to $\mathcal{P}_0$, which is equivalent to showing that $\mathrm{E}[Y] = 0$. This is not totally obvious since the X_i may repeat themselves, but it follows from the Isserlis formula and a combinatorial argument:

$$\begin{aligned}\mathrm{E}[Y] &= \sum_{\mathfrak{m}\oplus\mathfrak{u}} (-1)^{|\mathfrak{m}|} \prod_{(i,j)\in\mathfrak{m}} \Gamma_{i,j}\,\mathrm{E}\left[\prod_{k\in\mathfrak{u}} X_k\right]\\ &= \sum_{\mathfrak{m}\oplus\mathfrak{u}} (-1)^{|\mathfrak{m}|} \prod_{(i,j)\in\mathfrak{m}} \Gamma_{i,j} \sum_{\substack{\text{perfect matchings}\\ \mathfrak{m}'(\mathfrak{u})\text{ of }\mathfrak{u}}} \prod_{(i',j')\in\mathfrak{m}'} \Gamma_{i',j'}\\ &= \sum_{\mathfrak{m}\oplus\mathfrak{u}} \sum_{\mathfrak{m}'(\mathfrak{u})} (-1)^{|\mathfrak{m}|} \prod_{(i,j)\in\mathfrak{m}} \Gamma_{i,j} \prod_{(i',j')\in\mathfrak{m}'(\mathfrak{u})} \Gamma_{i',j'}\\ &= \sum_{\mathfrak{m}([n])} \prod_{(i,j)\in\mathfrak{m}} \Gamma_{i,j} \sum_{\mathfrak{m}'\subset\mathfrak{m}} (-1)^{|\mathfrak{m}'|}.\end{aligned}$$

The second equality is simply the Isserlis formula and the third is simply a rearrangment of the sum. The final equality is the most important. Summing over $\mathfrak{m} \oplus \mathfrak{u}$ and $\mathfrak{m}'(\mathfrak{u})$ after the third equality is equivalent to summing over all perfect matchings of $[n] = \{1, \dots, n\}$, which is how we arrive to the fourth equality. The product terms after the third equality clearly only depend on the full perfect matching, while the term $(-1)^{|\mathfrak{m}|}$ depends on a subset of the perfect matching, which is why we isolate it and sum over all such subsets in the last equality. Now from the distributive property of multiplication it follows that for any finite subset A

$$(1 + (-1))^{|A|} = \sum_{B \subset A} (-1)^{|B|} = 0,$$

thus by using this property above we get $\mathrm{E}[Y] = 0$ as claimed.

Now to show that Y is orthogonal to general polynomials of degree $< n$ is not much different. By linearity it is enough to show that $\mathrm{E}[Y X_{n+1} \dots X_{n+\ell}] = 0$ for any Gaussian variables $X_{n+1}, \dots, X_{n+\ell}$, where $\ell \leqslant n - 1$. Then we modify the above argument as follows:

$$\begin{aligned}
\mathrm{E}[Y X_{n+1} \dots X_{n+\ell}] &= \sum_{\mathfrak{m} \oplus \mathfrak{u}} (-1)^{|\mathfrak{m}|} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j}\, \mathrm{E}\left[X_{n+1} \dots X_{n+\ell} \prod_{k \in \mathfrak{u}} X_k \right] \\
&= \sum_{\mathfrak{m} \oplus \mathfrak{u}} (-1)^{|\mathfrak{m}|} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j} \sum_{\substack{\text{perfect matchings} \\ \mathfrak{m}' \text{ of } \mathfrak{u} \cup \{n+1, \dots, n+\ell\}}} \prod_{(i',j') \in \mathfrak{m}'} \Gamma_{i',j'} \\
&= \sum_{\mathfrak{m}([n+\ell])} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j} \sum_{\mathfrak{m}'([n]) \subset \mathfrak{m}} (-1)^{|\mathfrak{m}'([n])|}.
\end{aligned}$$

Note that the second line can be rearranged into a sum over all perfect matchings of $[n+\ell] = \{1, \dots, n+\ell\}$, which is what happens in the third line. However since the term $(-1)^{|\mathfrak{m}|}$ in the second line involves only matchings of $[n] = \{1, \dots, n\}$, when we rearrange the sums the inner summation is only over perfect matchings $\mathfrak{m}'$ of $\{1, \dots, n\}$ that are subsets of perfect matchings $\mathfrak{m}$ of $\{1, \dots, n+\ell\}$. For the same reasons as in the previous argument for $\mathcal{P}_0$ the inner summation is zero in most cases, but there is the special case in which a perfect matching $\mathfrak{m}$ of $[n+\ell]$ does not contain any perfect matchings of $[n]$. In that case the inner sum would be identically one. However, for $\ell \leqslant n - 1$ that case is impossible, because it would require matching every element of $[n]$ with an element in $\{n+1, \dots, n+\ell\}$, and with $\ell \leqslant n-1$ that cannot be done (pigenhole principle). Thus we conclude that $\mathrm{E}[Y X_{n+1} \dots X_{n+\ell}] = 0$, completing the proof that $Y \in \mathcal{P}_n$.

Now we need to show that $Y = \pi_n(X_1 \dots X_n)$. Recall that $\pi_n(X_1 \dots X_n)$ is the unique element of $\mathcal{P}_n$ that is orthogonal to $X_1 \dots X_n - \pi_n(X_1 \dots X_n)$. Since $Y \in \mathcal{P}_n$ it is enough to show that $X_1 \dots X_n - Y$ is orthogonal to Y. But this actually follows from $Y \in \mathcal{P}_n$, since (3.9) implies that $X_1 \dots X_n - Y$ is a polynomial of degree $n-2$. Indeed, the only term of degree $\geqslant n-1$ in Y is $X_1 \dots X_n$ itself, corresponding to the partial matching in which all elements are left unmatched. Thus Y satisfies the properties required to be the projection, so by uniqueness we conclude that $Y = \pi_n(X_1 \dots X_n)$.

Finally, the decomposition formula (3.10) follows by using (3.9) and similar combinatorics as above. Insert (3.9) into the right hand side of (3.10) to obtain

$$\sum_{\substack{\mathfrak{m} \oplus \mathfrak{u} \\ \text{of } [n]}} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j}\, \pi_{|\mathfrak{u}|} \left(\prod_{k \in \mathfrak{u}} X_k \right) = \sum_{\substack{\mathfrak{m} \oplus \mathfrak{u} \\ \text{of } [n]}} \sum_{\substack{\mathfrak{m}' \oplus \mathfrak{u}' \\ \text{of } \mathfrak{u}}} (-1)^{|\mathfrak{m}'|} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j} \prod_{(i',j') \in \mathfrak{m}'} \Gamma_{i',j'} \prod_{k \in \mathfrak{u}'} X_k.$$

Every term in the right hand side's double summation clearly corresponds to a partial matching of $[n] = \{1, \ldots, n\}$ (namely to $(\mathfrak{m} \cup \mathfrak{m}') \oplus \mathfrak{u}'$) and the only term in the summand that does not depend on this partial matching (i.e. does not depend on $\mathfrak{m} \cup \mathfrak{m}'$ or $\mathfrak{u}'$) is $(-1)^{|\mathfrak{m}'|}$. Thus the right hand side can be rewritten as

$$\sum_{\substack{\mathfrak{m} \oplus \mathfrak{u} \\ \text{of } [n]}} \prod_{(i,j) \in \mathfrak{m}} \Gamma_{i,j} \prod_{k \in \mathfrak{u}} X_k \sum_{\mathfrak{m}' \subset \mathfrak{m}} (-1)^{|\mathfrak{m}'|}.$$

As in the previous cases all terms in the alternating sum cancel, except for when $\mathfrak{m} = \emptyset$ when it trivially gives a value of one. Thus the entire sum collapses to just one term with $\mathfrak{m} = \emptyset$, which happens iff all elements are unmatched, i.e. iff $\mathfrak{u} = [n]$. Thus all that remains is $X_1 \ldots X_n$, thereby completing the proof of (3.10). $\square$

Problems

1. Prove that if we apply the Gram-Schmidt orthogonalization procedure – in Gauss space – to the monomials $\{1, x, x^2, x^3, \ldots\}$, then we obtain the Hermite polynomials $H_1, H_2, \ldots$.

⟨ex:Hermite:Taylor⟩

2. Choose and fix an $w \in \mathbb{R}$ and define $f(z) = \exp(wz - w^2/2)$ for all $z \in \mathbb{R}$.
   (a) Verify that $f \in \mathbb{D}^{1,2}(\mathrm{P}_n)$ and calculate $\|f\|_{1,2}$.
   (b) Use integration by parts to show that $\mathrm{E}[f(Z)H_k] = w^k$.
   (c) Conclude that $H_k(x)$ is the coefficient of $w^k/k!$ in the Taylor series expansion of $w \mapsto \exp(wx - w^2/2)$. In other words, the Hermite polynomials are defined uniquely via the relation,
$$\exp\left(wx - \frac{w^2}{2}\right) = \sum_{k=0}^{\infty} \frac{w^k}{k!} H_k(x).$$
3. Verify that $H_n(0) = 0$ for all odd integers n, and
$$H_n(0) = (-1)^{n-1}(n-1)!! \quad \text{for all even integers } n,$$
   where $k!! := k \times (k-2) \times (k-4) \times \cdots \times 1$ for all even integers k.
4. Recall the adjoint operators A_j of (2.4). Show that for $k \in \mathbb{Z}_+^n$
$$A_1^{k_1} \ldots A_n^{k_n} 1 = \mathcal{H}_k(Z),$$
   where A_j^0 is the identity operator. Show that the order of the adjoint operators also doesn't matter, so that if $q \in \{1, \ldots, n\}^p$ and if σ is a permutation of $\{1, 2, \ldots, p\}$ then
$$A_{q_1} \ldots A_{q_p} 1 = A_{q_{\sigma(1)}} \ldots A_{q_{\sigma(p)}} 1.$$
5. Derive the following Hermite-function version of the binomial theorem:
$$H_n(a + x) = \sum_{k=0}^{n} \binom{n}{k} a^k H_{n-k}(x) \quad \text{for all } a, x \in \mathbb{R} \text{ and } n \in \mathbb{Z}_+.$$
6. Use Wick's formula (3.9) and the fact that $\pi_n(Z^n) = H_n(Z)$ to show that the Hermite polynomials can be written as
$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{2^k k!(n-2k)!} x^{n-2k}.$$
7. Use Wick's formula (3.10) to show that
$$x^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k k!} H_{n-2k}(x).$$
   Then re-verify this formula using induction and $H_{k+1}(x) = xH_k(x) - kH_{k-1}(x)$.
8. (**This problem requires some background in Itô calculus.**)
   Let B be a standard Brownian motion, and define $B_n(t) := \int_0^t B_{n-1}(s)\,\mathrm{d}B(s)$ as an Itô integral for every integer $n \geqslant 1$, where $B_0(t) := 1$ for all $t \geqslant 0$. These are *multiple Itô integrals*; for example,
$$B_1(t) = B(t), \quad B_2(t) = \int_0^t \int_0^s \mathrm{d}B(r)\,\mathrm{d}B(s), \cdots$$
$$B_{n+1}(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_n} \mathrm{d}B(s_{n+1}) \cdots \mathrm{d}B(s_2)\,\mathrm{d}B(s_1), \cdots .$$

Choose and fix some $\alpha \geqslant 0$, and define $X(t) := \sum_{n=0}^{\infty} \alpha^n B_n(t)$ for all $t \geqslant 0$.

(a) Verify that for every $T > 0$, the series converges in $L^2(\Omega)$, uniformly for $t \in [0\,,T]$. [Hint: Doob's maximal $L^2(\Omega)$ inequality.]

(b) Prove that X satisfies the Itô stochastic differential equation, $\mathrm{d}X(t) = \alpha X(t)\mathrm{d}B(t)$ subject to $X(0) = 1$. Conclude that

$$X(t) = \exp\left(\alpha B(t) - \frac{t\alpha^2}{2}\right) \qquad \text{for all } t \geqslant 0 \text{ a.s.}$$

(c) Compare (b) to Problem 2 in order to conclude that

$$H_n(B(1)) = n! \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \mathrm{d}B(s_n) \cdots \mathrm{d}B(s_2)\, \mathrm{d}B(s_1) \quad \text{a.s.}$$

Because $B(1)$ has the same distribution as Z, the above gives a particular construction of $H_n(Z)$ using Brownian motion. This construction is part of a deep theory of Wiener XXX and Itô XXX. The exposition is due to McKean XXX.

9. Extend Problem 2 to $\mathbb{R}^n$ for all $n \geqslant 1$ by showing that

$$\exp\left(w \cdot x - \frac{\|w\|^2}{2}\right) = \sum_{k \in \mathbb{Z}_+^n} \frac{w^k}{k!}\, \mathcal{H}_k(x) \qquad \text{for every } w, x \in \mathbb{R}^n,$$

where $w^k := \prod_{i=1}^n w_i^{k_i}$.

⟨ex:Dk,2⟩ 10. Suppose that $f \in L^2(\mathrm{P}_n)$ satisfies

$$\sum_{k \in \mathbb{Z}_+^n} \frac{\|k\|^{2m}}{k!} \left|\mathrm{E}(f\mathcal{H}_k)\right|^2 < \infty \qquad \text{for some } m \in \mathbb{N}.$$

Prove that $f \in \mathbb{D}^{m,2}(\mathrm{P}_n)$, using the following steps:

(a) Expand f in terms of Hermite polynomials, and let f_ℓ denote the same sum but restricted to indices $k \in \mathbb{Z}_+^n$ that satisfy $\|k\| \leqslant \ell$. Prove that $f_\ell \in C_0^\infty(\mathrm{P}_n)$ and $\lim_{\ell\to\infty} f_\ell = f$ in $L^2(\mathrm{P}_n)$. Use this to deduce that it suffices to prove that $\{D_{i_1} \cdots D_{i_m} f_\ell\}_{\ell=1}^\infty$ is Cauchy in $L^2(\mathrm{P}_n)$ for every $i_1, \ldots, i_m \in \{1\,, \ldots, n\}$.

(b) Now find an expression for $D_{i_1} \cdots D_{i_m} f_\ell$ in terms of Hermite polynomials. [Hint: Examine the proof of Proposition 2.4.]

11. Show that if we are working in $L^2(\mathrm{P}_n)$ then the jth subspace/Wiener chaos $\mathcal{P}_j$ has dimension

$$\binom{j+n-1}{n-1}.$$

⟨que:quadratic_form_Wick⟩ 12. Suppose we are working in $L^2(\mathrm{P}_n)$, so that $Z_1, \ldots, Z_n$ are iid standard normals. As usual write $Z = (Z_1, \ldots, Z_n)'$ for the random column vector consisting of these normals (recall $'$ means transpose). Let A be an $n \times n$ matrix of real numbers.

(a) Show that the quadratic form $Z'AZ$ is unchanged if A is replaced by its symmetrized version $(A + A')/2$.

(b) Thus assume from now on that $A = A'$, i.e. A is symmetric. Show that $\pi_2(Z'AZ) = Z'AZ - \mathrm{tr}(A)$, where $\mathrm{tr}(A)$ is the trace of the matrix A.

(c) More generally, show that if we are working under $L^2(\mathrm{Q}_n)$ where Q_n is the measure of the $\mathrm{N}_n(0,\Sigma)$ distribution, then $\pi_2(Z'AZ) = Z'AZ - \mathrm{tr}(A\Sigma)$.

(d) Finally, show the following higher degree version. For an integer $k \geqslant 2$ let K be a k-tensor, meaning that $K : [n]^k \to \mathbb{R}$, and for the polynomial of degree k given by

$$f = \sum_{q\in[n]^k} K_q Z_{q_1} \dots Z_{q_k}$$

show that under $L^2(\mathrm{P}_n)$

$$\pi_k(f) = \sum_{q\in[n]^k} K_q \pi_k(Z_{q_1} \dots Z_{q_k}) = \sum_{q\in[n]^k} K_q \mathcal{H}_{c(q)}(Z)$$

where $c(q) = (c_1(q), \dots, c_n(q))$ and $c_i(q)$ is the number of times that i appears in the k-tuple q, for $i \in [n] = \{1, \dots, n\}$. Further simplify this formula by showing that every such f can be represented by the symmetrized version $\tilde{K}$ of the k-tensor as

$$f = \sum_{q\in[n]^k} \tilde{K}_q Z_{q_1} \dots Z_{q_k}$$

where $\tilde{K}$ is defined by

$$\tilde{K}_q = \frac{c(q)!}{k!} \sum_{\sigma} K_q.$$

Here σ is a permutation of $\{1, \dots, k\}$ and $\sigma(q) = (q_{\sigma(1)}, \dots, q_{\sigma(k)})$, and the coefficient of the sum is the inverse of a multinomial coefficient. Conclude that

$$\pi_k(f) = \sum_{q\in[n]^k} \tilde{K}_q \mathcal{H}_{c(q)}(Z) = \sum_{q\in[n]^{\odot k}} \frac{k!}{c(q)!} \tilde{K}_q \mathcal{H}_{c(q)}(Z),$$

where $[n]^{\odot k} = [n]^k/\sim$, and $\sim$ is the equivalence relation $q \sim q'$ iff $c(q) = c(q')$, i.e. each element of $[n]$ appears the same number of times in both q and q' and therefore one is just a permutation of the other.

13. Prove that the Poincaré inequality on $\mathbb{R}^n$ [Proposition 2.4] follows directly from the one-dimensional case [Corollary 1.6] and induction on the value of $n \geqslant 1$. This method is sometimes called "tensorization."

14. Let $f(x) := \max_{1\leqslant i\leqslant n} x_i$ for all $x \in \mathbb{R}^n$, and prove that Proposition 2.9 improves Proposition 2.8. That is, prove that $\lambda_* \,\mathrm{E}(\|Df\|^2) \geqslant \max_{1\leqslant i\leqslant n} \mathrm{Var}(X_i)$ for the present choice of f.

The following Problems 15–20 depend sequentially on one another. Throughout these problems, let us choose and fix some $\lambda > 0$, and let X have a Poisson distribution with $\mathrm{E}(X) = \lambda$. Also, let μ denote the distribution of X; that is, $\mu\{k\} = \mathrm{e}^{-\lambda}\lambda^k/k!$ for $k \in \mathbb{Z}_+$ and $\mu\{k\} = 0$ otherwise. Finally, define $C_0, C_1, C_2, \dots$ canonically as the real-valued functions on $\mathbb{Z}_+$ that satisfy the following for all $x = 0, 1, 2, \dots$ and $w > -1$:

$$\mathrm{e}^{-w\lambda}(1+w)^x = \sum_{k=0}^{\infty} \frac{w^k}{k!}\, C_k(x).$$

Many authors usual write $C_k^{(\lambda)}$ instead of C_k, and refers to C_k as the kth *monic Charlier polynomial* with parameter λ.

⟨ex:Poisson:1⟩ 15. Prove that $C_0(x) = 1$ and $C_k(0) = (-\lambda)^k$ for all $x, k \in \mathbb{Z}_+$.

16. Verify that $\mathrm{E}[\exp\{-w\lambda\}(1+w)^X] = 1$ for all $w > -1$. Conclude from this that $\mathrm{E}[C_k(X)] = 0$ for all $k \geqslant 1$.

17. Verify that

$$C_k(x) = \sum_{m=0}^{k \wedge x} \binom{k}{m}\binom{x}{m} m!(-\lambda)^{k-m} \quad \text{for all } x, k \in \mathbb{Z}_+,$$

and conclude that every C_k is a polynomial of degree at most k on the semigroup $\mathbb{Z}_+$.

18. Prove that the sequence $\left\{\sqrt{n!/\lambda^n}\, C_k\right\}_{k=0}^{\infty}$ is a complete orthonormal basis for $L^2(\mu)$. Conclude that for all $f, g \in L^2(\mu)$,

$$\mathrm{Cov}[f(X)\,, g(X)] = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \langle f\,, C_k\rangle_{L^2(\mu)} \langle g\,, C_k\rangle_{L^2(\mu)}.$$

[Hint: Consider the second moment of $\exp\{-w\lambda\}(1+w)^X$.]

⟨ex:Poisson:n-1⟩ 19. Define a linear operator $\mathscr{A}$ via the following:

$$(\mathscr{A} f)(x) := xf(x-1) - \lambda f(x) \qquad \text{for every } f : \mathbb{Z}_+ \to \mathbb{R} \text{ and } x \in \mathbb{Z}_+,$$

where $f(-1) := 0$. Show that $\mathscr{A}$ is a linear mapping from $L^2(\mu)$ to $L^2(\mu)$ and whose adjoint is $\mathscr{D}$, where $(\mathscr{D} f)(x) := \lambda\{f(x+1) - f(x)\}$. Then proceed to verify the following, steps which essentially show that the role of the pair $(\mathscr{D}\,, \mathscr{A})$ is the "Poisson space" analogue of the role of the pair $(D\,, A)$ in the Gauss space:

(a) Prove that $C_{k+1} = \mathscr{A} C_k$ for all $k \geqslant 0$. [Hint: Start with the derivative of $w \mapsto \mathrm{e}^{-w\lambda}(1+w)^x$.]

(b) Prove that $\mathscr{D} C_{k+1} = (k+1)C_k$ for all $k \in \mathbb{Z}_+$.

(c) Prove that $\mathrm{Var}[f(X)] \leqslant \lambda\, \mathrm{E}(|\mathscr{D} f)(X)|^2)$ for every $f \in L^2(\mu)$.

⟨ex:Poisson:n⟩ 20. Use Problem 19 and the central limit theorem in order to find another proof of the Poincaré inequality for P_1 [Corollary 1.6].