Math 6070-1, Spring 2014;
Partial Solutions to Assignment #2

1. Suppose $X_1, \ldots, X_n$ is a random [that is, i.i.d.] sample from a Uniform$(\theta\,, 2\theta)$ distribution, where $\theta > 0$ is an unknown parameter.

   (a) Find the maximum likelihood estimator for θ.

   Solution. The likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \theta^{-1}\mathbf{I}\left\{\theta < X_j < 2\theta\right\} = \theta^{-n}\mathbf{I}\left\{\frac{X_{n:n}}{2} \le \theta \le X_{1:n}\right\},$$

   where $X_{j:n}$ denotes the jth order statistic of $\{X_1\,, \ldots, X_n\}$. Because the function $h(\theta) := \theta^{-n}$ is decreasing, it follows that the MLE is

$$\hat\theta = \frac{X_{n:n}}{2} \quad \text{provided that} \quad \frac{X_{n:n}}{2} \le X_{1:n}.$$

   It remains to check that we always have $X_{n:n}/2 \le X_{1:n}$, so that the MLE is always uniquely equal to $X_{n:n}/2$. [This follows simply because $X_{n:n} \le 2\theta$ therefore, $X_{n:n}/2 \le \theta$, whereas $X_{1:n} \ge \theta$.]

   (b) Prove that the MLE is consistent.

   Solution. Recall that $\hat\theta = X_{n:n}/2$. If $x > 0$, then

$$\mathrm{P}\{\hat\theta/\theta \le x\} = \left(\mathrm{P}\{X_1 \le 2\theta x\}\right)^n.$$

   [If $x < 0$ then $\mathrm{P}\{\hat\theta/\theta \le x\} = 0$, vacuously.] The last displayed probability is equal to one if $2\theta x \ge 2\theta$ [i.e., $x \ge 1$] and zero if $2\theta x \le \theta$ [i.e., $x \le 1/2$]. On the other hand, if $\frac{1}{2} < x < 1$, then it follows that

$$\mathrm{P}\{\hat\theta/\theta \le x\} = \left(\int_\theta^{2\theta x} \theta^{-1}\,\mathrm{d}y\right)^n = (2x-1)^n.$$

   In conclusion, the pdf of $\hat\theta/\theta$ is

$$f_{\hat\theta/\theta}(x) = \begin{cases} 2n(2x-1)^{n-1} & \text{if } \frac{1}{2} < x < 1,\\ 0 & \text{otherwise.}\end{cases}$$

We can use this to compute

$$
\begin{aligned}
\mathrm{E}[\hat{\theta}/\theta] &= 2n \int_{1/2}^{1} x(2x-1)^{n-1}\,\mathrm{d}x \\
&= n \int_0^1 \left(\frac{y+1}{2}\right) y^{n-1}\,\mathrm{d}y \qquad [y := 2x-1] \\
&= \frac{n}{2}\left[\int_0^1 y^n\,\mathrm{d}y + \int_0^1 y^{n-1}\,\mathrm{d}y\right] \\
&= \frac{n}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) = 1 - \frac{1}{2(n+1)}.
\end{aligned} \tag{1}
$$

Also,

$$
\begin{aligned}
\mathrm{E}\left(|\hat{\theta}/\theta|^2\right) &= 2n \int_{1/2}^{1} x^2(2x-1)^{n-1}\,\mathrm{d}x \\
&= n \int_0^1 \left(\frac{y+1}{2}\right)^2 y^{n-1}\,\mathrm{d}y \qquad [y := 2x-1] \\
&= \frac{n}{4} \int_0^1 (y^2+2y+1) y^{n-1}\,\mathrm{d}y \\
&= \frac{n}{4}\left[\int_0^1 y^{n+1}\,\mathrm{d}y + 2\int_0^1 y^n\,\mathrm{d}y + \int_0^1 y^{n-1}\,\mathrm{d}y\right] \\
&= \frac{n}{4}\left[\frac{1}{n+2} + \frac{2}{n+1} + \frac{1}{n}\right].
\end{aligned}
$$

Note that $(n+1)^{-1} = n^{-1} - a_n$ where $a_n \approx n^{-2}$ as $n \to \infty$. Similarly, $(n+2)^{-1} = n^{-1} - b_n$ where $b_n \approx 2n^{-2}$ as $n \to \infty$. Therefore,

$$
\mathrm{E}\left(|\hat{\theta}/\theta|^2\right) = \frac{n}{4}\left[\frac{4}{n} - a_n - b_n\right] = 1 - c_n,
$$

where $c_n = \frac{1}{4}\{na_n + nb_n\} \approx 3/(4n) \to 0$ as $n \to \infty$. This fact and (1) together yield

$$
\begin{aligned}
\mathrm{Var}(\hat{\theta}/\theta) &= 1 - c_n - \left[1 - \frac{1}{2(n+1)}\right]^2 \\
&= -c_n + \frac{1}{n+1} - \frac{1}{4(n+1)^2} \to 0,
\end{aligned}
$$

as $n \to \infty$. Therefore, the Chebyshev inequality yields

$$
\frac{\hat{\theta}}{\theta} - \mathrm{E}\left[\frac{\hat{\theta}}{\theta}\right] \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty.
$$

One more application of (1) shows that $\hat{\theta}/\theta \overset{\mathrm{P}}{\to} 1$ as $n \to \infty$. This is equivalent to the desired consistency of $\hat{\theta}$.

(c) Calculate the bias of the MLE. Use your computation to verify that the MLE is asymptotically unbiased [that is, show that $\mathrm{bias}(\hat{\theta}) \to 0$ as $n \to \infty$].

Solution. We saw earlier in (1) that $\mathrm{E}(\hat{\theta}/\theta) = 1 - \{2(n+1)\}^{-1}$. Therefore,

$$\mathrm{Bias}(\hat{\theta}) = \theta - \mathrm{E}(\hat{\theta}) = \frac{\theta}{2(n+1)}.$$

Clearly, this quantity converges to zero as $n \to \infty$.