Math 6070-1, Spring 2014
Partial Solutions to Assignment #1

1. Let X denote a random variable with a socalled "double exponential" density function,

$$f(x) = \tfrac{1}{2}\mathrm{e}^{-|x|} \qquad (-\infty < x < \infty).$$

(a) Compute the moment generating function and the characteristic function of X.

Solution. The moment generating function is $M(t) = \int_{-\infty}^{\infty} \mathrm{e}^{tx} f(x)\,\mathrm{d}x$. Therefore, we may write

$$\begin{aligned} M(t) &= \frac{1}{2}\int_{-\infty}^{0} \mathrm{e}^{(1+t)x}\,\mathrm{d}x + \frac{1}{2}\int_{0}^{\infty} \mathrm{e}^{(t-1)x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{0}^{\infty} \mathrm{e}^{-(1+t)x}\,\mathrm{d}x + \frac{1}{2}\int_{0}^{\infty} \mathrm{e}^{(t-1)x}\,\mathrm{d}x. \end{aligned}$$

If $t \geq 1$, then the second integral diverges, whereas the first diverges if $t \leq -1$. Therefore,

$$M(t) = \infty \quad \text{if } |t| \geq 1.$$

On the other hand, if $|t| < 1$, then both of the said integrals converge, and

$$M(t) = \frac{1}{2(1+t)} + \frac{1}{2(1-t)} = \frac{2}{1-t^2} \quad \text{if } |t| < 1.$$

The characteristic function is computed by formally setting $C(t) = M(it)$; we obtain $C(t) = 2(1+t^2)^{-1}$ for all real numbers t. [Why not just $|t| \leq 1$?]

(b) Use your computations to evaluate $\mathrm{E}(X^n)$ for every integer $n \geq 1$. Justify your method.

Solution. Since $M^{(n)}$ exists in $(-1\,,1)$, it follows that $\mathrm{E}(X^n) = M^{(n)}(0)$. Thus, for example,

$$\mathrm{E}(X) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{1-t^2}\right)\Bigg|_{t=0} = 0.$$

It is not so easy to compute $M^{(n)}$ directly. But the moments are not hard to find:

$$\mathrm{E}(X^n) = \tfrac{1}{2}\int_{-\infty}^{\infty} x^n \mathrm{e}^{-|x|}\,\mathrm{d}x.$$

We are integrating an odd function when n is an odd integer. Therefore, $\mathrm{E}(X^n) = 0$ when n is odd. When n is even, however, we obtain by symmetry,

$$\mathrm{E}(X^n) = \int_0^{\infty} x^n \mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(n+1) = n!.$$

2. Prove that if $X_1, X_2, \ldots, X_n$ form an i.i.d. sample from a Uniform$(0\,,1)$ distribution, then $\prod_{j=1}^n X_j^{1/n}$ converges to $1/\mathrm{e}$ in probability as $n \to \infty$.

Solution. Since $0 < X_j < 1$ for all $j \geq 1$, we can define unambiguously $Y_j := \ln X_j$. Then the Y_j's are i.i.d., negative, and

$$F_{Y_1}(x) = \mathrm{P}\{Y_1 \leq x\} = \mathrm{P}\{X_1 \leq \mathrm{e}^x\} = \mathrm{e}^x \qquad \text{for all } x \leq 0.$$

That is, the pdf of Y_1 is $f_{Y_1}(x) = \mathrm{e}^x I\{x \leq 0\}$, which tells us that the $-Y_j$'s are i.i.d. Exponential(1) random variables. Since $\mathrm{E}(Y_1) = -1$, it follows that

$$\prod_{j=1}^n X_j^{1/n} = \mathrm{e}^{\bar{Y}_n} \xrightarrow{\mathrm{P}} \mathrm{e}^{-1} \qquad \text{as } n \to \infty,$$

thanks to the law of large numbers.

3. Suppose $X_1, X_2, \ldots$ are i.i.d. random variables, selected from a Poisson(1) distribution. Define $S_n := X_1 + \cdots + X_n$ for every $n \geq 1$.

   (a) Compute the distribution of S_n for every $n \geq 1$.

   Solution. The MGF of X_1 is $M_{X_1}(t) = \exp\{\mathrm{e}^t - 1\}$. Therefore, the MGF of S_n is

   $$M_{S_n}(t) = \prod_{j=1}^n M_{X_j}(t) = \exp\left\{n\left(\mathrm{e}^t - 1\right)\right\}.$$

   This is the MGF of a Poisson(n); therefore, $S_n \sim$ Poisson(n).

   (b) Use the central limit theorem to approximate $\mathrm{P}\{S_{100} \leq 120\}$.

   Solution. Because $\mu = \mathrm{E}(X_1) = 1$ and $\sigma^2 = \mathrm{Var}(X_1) = 1$, the central limit theorem states that $\mathrm{P}\{S_n \leq n + x\sqrt{n}\} \approx \mathrm{P}\{N(n\,,n) \leq n + x\sqrt{n}\} = \Phi(x)$ for every x, as long as n is large. Set $x = 2$ to see that

   $$\mathrm{P}\left\{S_{100} \leq 120\right\} \approx \Phi(2) \approx 0.9772,$$

   provided that $n = 100$ is large enough to allow an appeal to the CLT.