Math 6070-1, Spring 2006
Solutions to Homework 1

1. *Compute, carefully, the moment generating function of a Gamma$(\alpha\,,\beta)$. Use it to compute the moments of a Gamma-distributed random variable.*

   Solution: Recall that
   $$f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x},\qquad 0<x<\infty.$$
   Therefore,
   $$M(t)=\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}\mathrm{e}^{-(\beta-t)x}\,\mathrm{d}x.$$
   If $t\geq\beta$, then $M(t)=\infty$. But if $t<\beta$ then $M(t)=\beta^\alpha/(\beta-t)^\alpha$. Its derivatives are $M'(t)=\beta^\alpha\alpha(\beta-t)^{-\alpha-1}$, $M''(t)=\beta^\alpha\alpha(\alpha-1)(\beta-t)^{-\alpha-2}$, and so on. The general term is $M^{(k)}(t)=\beta^\alpha(\beta-t)^{-\alpha-k}\prod_{\ell=0}^{k-1}(\alpha-\ell)$. Thus, $\mathrm{E}X=M'(0)=\alpha/\beta$, $\mathrm{E}(X^2)=M''(0)=\alpha(\alpha-1)/\beta^2$, and so. In general, we have $\mathrm{E}(X^k)=\beta^{-k}\prod_{j=0}^{k-1}(\alpha-j)$.

2. *Let $X_1,X_2,\ldots,X_n$ be an independent sample (i.e., they are i.i.d.) with finite mean $\mu=\mathrm{E}X_1$ and variance $\sigma^2=\mathrm{Var}X_1$. Define*
   $$\hat\sigma_n^2:=\frac{1}{n}\sum_{j=1}^n\left(X_j-\bar X_n\right)^2, \tag{1}$$
   where $\bar X_n:=(X_1+\cdots+X_n)/n$ denotes the sample average. First, compute $\mathrm{E}(\hat\sigma_n^2)$. Then prove, carefully, that $\hat\sigma_n^2$ converges in probability to σ^2.

   Solution: $\hat\sigma_n^2=n^{-1}\sum_{j=1}^n X_j^2-(\bar X_n)^2$. Therefore, $\mathrm{E}\hat\sigma_n^2=\mathrm{E}[X_1^2]-\mathrm{E}[(\bar X_n)^2]$. But $\mathrm{E}X_1^2=\mathrm{Var}(X_1)+\mu^2=\sigma^2+\mu^2$. Similarly, $\mathrm{E}(\bar X_n^2)=\mathrm{Var}(\bar X_n)+(\mathrm{E}\bar X_n)^2=(\sigma^2/n)+\mu^2$. Therefore, $\mathrm{E}(\hat\sigma_n^2)=\sigma^2(n-1)/n$. As regards the large-sample theory, we apply the law of large numbers twice: $\bar X_n\xrightarrow{\mathrm{P}}\mu$; and $n^{-1}\sum_{=1}^n X_i^2\xrightarrow{\mathrm{P}}\sigma^2+\mu^2$. This proves that $\hat\sigma_n^2$ is a consistent estimator of σ^2.

3. *Let U have the Uniform-$(0\,,\pi)$ distribution.*

   (a) *Prove that if F is a distribution function and F^{-1}—its inverse function—exists, then the distribution function of $X:=F^{-1}(U)$ is F.*

Solution: $\mathrm{P}\{X \le x\} = \mathrm{P}\{U \le F(x)\} = F(x)$, whence we have $F_X = F$, as desired.

(b) *Use the preceding to prove that $X := \tan U$ has the Cauchy distribution. That is, the density function of Y is*

$$f_X(a) := \frac{1}{\pi(1+a^2)}, \qquad -\infty < a < \infty. \tag{2}$$

Solution: Let C have the Cauchy distribution. Then,

$$\begin{aligned} F_C(a) = \mathrm{P}\{C \le a\} &= \int_{-\infty}^{a} \frac{\mathrm{d}u}{\pi(1+u^2)} \\ &= \frac{1}{\pi}\arctan a - \frac{1}{\pi}\arctan(-\infty) = \frac{1}{\pi}\arctan(a) + \frac{1}{2}. \end{aligned}$$

Therefore, $F_C^{-1}(a) = \arctan(\pi x - \frac{\pi}{2})$, and $F_C^{-1}(U) \sim$ Cauchy. Note that $V := \pi U - (\pi/2) \sim$ Uniform-$(-\pi/2\,,\pi/2)$. So $\arctan(V)$ is indeed Cauchy, where $V \sim$ Uniform-$(-\pi/2\,,\pi/2)$.

(c) *Use the preceding to find a function h such that $Y := h(U)$ has the Exponential (λ) distribution.*

Solution: If $X \sim \mathrm{Exp}(\lambda)$ then $F(x) = \int_{-\infty}^{x} \lambda \mathrm{e}^{-\lambda z}\,\mathrm{d}z = 1 - \mathrm{e}^{-\lambda x}$. Thus, $h(x) := F^{-1}(x) = -\lambda^{-1}\ln(1-x)$, and $F^{-1}(U) \sim \mathrm{Exp}(\lambda)$. Note that $1-U$ has the same distribution as U. So $-\lambda^{-1}\ln(U) \sim$ Exponential(λ).

4. *A random variable X has the* logistic *distribution if its density function is*

$$f(x) = \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}, \qquad -\infty < x < \infty. \tag{3}$$

(a) *Compute the distribution function of X.*

Solution: Set $y := 1 + \mathrm{e}^{-x}$ to find that that

$$\begin{aligned} F(a) &= \int_{-\infty}^{a} \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}\,\mathrm{d}x = \int_{1+\exp(-a)}^{\infty} \frac{\mathrm{d}y}{y^2} \\ &= \frac{1}{1+\mathrm{e}^{-a}}, \qquad -\infty < a < \infty. \end{aligned}$$

(b) *Compute the moment generating function of X.*

Solution: Again set $y := \mathrm{e}^{-x}$ to find that

$$M(t) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{(t-1)x}}{(1+\mathrm{e}^{-x})^2}\,\mathrm{d}x = \int_{0}^{\infty} \frac{y^{-t}}{(1+y)^2}\,\mathrm{d}y.$$

If $|t| \ge 1$ then this is ∞ [consider the integral for small y when $t \ge 1$, or large y when $t \le -1$]. On the other hand, if $|t| < 1$ then this is finite. Remarkably enough, $M(t)$ can be computed explicitly when $|t| < 1$. Note that

$$\frac{1}{(1+y)^2} = \int_0^\infty z\mathrm{e}^{-z(1+y)}\,\mathrm{d}z.$$

Therefore,

$$\begin{aligned} M(t) &= \int_0^\infty \left(\int_0^\infty z\mathrm{e}^{-z(1+y)}\,\mathrm{d}z\right) y^{-t}\,\mathrm{d}y \\ &= \int_0^\infty \left(\int_0^\infty y^{-t}\mathrm{e}^{-zy}\,\mathrm{d}y\right) z\mathrm{e}^{-z}\,\mathrm{d}z, \end{aligned}$$

because the order of integration can be reversed. A change of variable [$w := yz$] shows that the inner integral is

$$\begin{aligned} \int_0^\infty y^{-t}\mathrm{e}^{-zy}\,\mathrm{d}y &= z^{t-1}\int_0^\infty w^{-t}\mathrm{e}^{-w}\,\mathrm{d}w \\ &= z^{t-1}\Gamma(1-t). \end{aligned}$$

Therefore, whenever $|t| < 1$,

$$\begin{aligned} M(t) &= \Gamma(1-t)\int_0^\infty z^t\mathrm{e}^{-z}\,\mathrm{d}z \\ &= \Gamma(1-t)\Gamma(1+t). \end{aligned}$$

There are other ways of deriving this identity as well. For instance, we can write $w := (1+y)^{-1}$, so that $\mathrm{d}w = -(1+y)^{-2}\,\mathrm{d}y$ and $y = (1/w) - 1$. Thus,

$$\begin{aligned} M(t) &= \int_0^1 \left(\frac{1}{w} - 1\right)^{-t}\,\mathrm{d}w \\ &= \int_0^1 w^t(1-w)^{-t}\,\mathrm{d}w = \mathrm{B}(1+t\,,1-t), \end{aligned}$$

where $\mathrm{B}(a\,,b) := \int_0^1 w^{a-1}(1-w)^{b-1}\,\mathrm{d}w$ denotes the "beta function." From tables (for instance), we know that $\mathrm{B}(a\,,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Therefore, $M(t) = \Gamma(1+t)\Gamma(1-t)/\Gamma(2) = \Gamma(1+t)\Gamma(1-t)$, as desired.

(c) *Prove that* $\mathrm{E}\{|X|^r\} < \infty$ *for all* $r > 0$.

Solution: We will prove a more general result: If $M(t) < \infty$ for all $t \in (-t_0\,,t_0)$ where $t_0 > 0$, then $\mathrm{E}(|X|^k) < \infty$ for all $k \geq 1$. Recall that $\mathrm{E}Z = \int_0^\infty \mathrm{P}\{Z > t\}\,\mathrm{d}t$ whenever $Z \geq 0$. Apply this with

$Z := |X|^k$ to find that

$$\begin{aligned}
\mathrm{E}(|X|^k) &= \int_0^\infty \mathrm{P}\left\{|X|^k > r\right\} \mathrm{d}r \\
&= \int_0^\infty \mathrm{P}\left\{|X| > r^{1/k}\right\} \mathrm{d}r \\
&= k \int_0^\infty \mathrm{P}\{|X| > s\} s^{k-1} \, \mathrm{d}s.
\end{aligned}$$

[Set $s := r^{1/k}$.] Fix $t \in (-t_0\,, t_0)$. Then, by Chebyshev's inequality,

$$\mathrm{P}\{|X| > s\} = \mathrm{P}\left\{\mathrm{e}^{t|X|} \geq \mathrm{e}^{ts}\right\} \leq \frac{\mathrm{E}[\mathrm{e}^{t|X|}]}{\mathrm{e}^{ts}}.$$

But $\mathrm{e}^{t|x|} \leq \mathrm{e}^{tx} + \mathrm{e}^{-tx}$ for all $x \in \mathbf{R}$, with room to spare. Therefore, $\mathrm{E}(r^{t|X|}) \leq M(t) + M(-t)$, whence it follows that for $t \in (-t_0\,, t_0)$ fixed and $A := M(t) + M(-t)$, $\mathrm{P}\{|X| > s\} \leq A\mathrm{e}^{-st}$. Thus,

$$\mathrm{E}(|X|^k) \leq Ak \int_0^\infty s^{k-1} \mathrm{e}^{-st} \, \mathrm{d}s = Akt^k \Gamma(k) = Ak!t^k < \infty.$$