Chapter 3

Measure Theory

3.2. Let $\Omega = \mathbf{N} = \{1, 2, \ldots\}$ denote the numerals. For all $n = 1, 2, \ldots$ define $\mathcal{F}_n$ to be the σ-algebra generated by $\{1\}, \ldots, \{n\}$. For example, $\mathcal{F}_1 = \{\varnothing, \mathbf{N}, \{1\}, \mathbf{N} \setminus \{1\}\}$, $\mathcal{F}_2 = \{\varnothing, \mathbf{N}, \{1\}, \{2\}, \{1, 2\}, \mathbf{N} \setminus \{1\}, \mathbf{N} \setminus \{2\}, \mathbf{N} \setminus \{1, 2\}\}$, and so on. Evidently, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. However, $\{1, 3, 5, \ldots\} \notin \cup_{i=1}^{\infty} \mathcal{F}_i$.

3.3. For this exercise, we need Cantor's countable axiom of choice: *A countable union of countable sets is itself countable.*

Let $\mathcal{F}$ denote the σ-algebra generated by all singletons in $\mathbf{R}$. Obviously, $\{x\} \in \mathcal{F}$ for all $x \in \mathbf{R}$. It remains to show that $[a, b] \notin \mathcal{F}$ for any $a \leqslant b$. Define $\mathcal{G} = \{G \subseteq \mathbf{R} : \text{ either } G \text{ or } G^c \text{ is denumerable}\}$. [Recall that "denumerable" means "at most countable."] You can check directly that $\mathcal{G}$ is a σ-algebra. We claim that $\mathcal{F} = \mathcal{G}$. This would prove that $[a, b] \notin \mathcal{F}$, because neither $[a, b]$ nor its complement are denumerable.

Let $\mathcal{A}$ denote the algebra generated by all singletons. If $E \in \mathcal{A}$, then either E is a finite collection of singletons, or E^c is. This proves that $\mathcal{A} \subseteq \mathcal{G}$. The monotone class theorem proves that $\sigma(\mathcal{A}) \subseteq \mathcal{G}$, but $\sigma(\mathcal{A}) = \mathcal{F}$. Therefore, it remains to prove that $\mathcal{G} \subseteq \mathcal{F}$. But it is manifest that $\mathcal{G} \subset \sigma(\mathcal{A})$.

3.4. $\mathcal{B}(\mathbf{R}^k)$ is the σ-algebra generated by all open sets. Let $\mathcal{O}$ define the σ-algebra generated by all open balls in $\mathbf{R}^k$ whose radius and center are rational. Because $\mathcal{O} \subseteq \mathcal{B}(\mathbf{R}^k)$, it remains to derive the converse inclusion. But general topology tells us that any open set is a countable union of open balls with rational centers and radii. This has the desired result.

3.9. If μ is a probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, then $F(a) := \mu(-\infty, a]$ is clearly non decreasing, $F(-\infty) = 0$, and $F(+\infty) = 1$. Finally, F is right-continuous, since μ is continuous from above [Lemma 3.11(b)]. Therefore, F is a distribution function.

Conversely, if F is a distribution function, then we may define $\mu(a, b] := F(b) - F(a)$, whenever $-\infty \leqslant a < b \leqslant \infty$. And if $(a_i, b_i]$ are disjoint for

$1 \leqslant i \leqslant N$, then

$$\mu\left(\bigcup_{i=1}^{N}(a_i\,,b_i]\right) := \sum_{i=1}^{N}\mu(a_i\,,b_i].$$

Appeal to telescoping sums in order to see that the preceding is a well-defined, rational definition. The preceding defines a finitely-additive measure on the algebra $\mathcal{A}$ of all finite disjoint unions of intervals of the form $(a\,,b]$. Thanks to the Carathéodory extension theorem, it remains to prove that μ is countably additive on $\mathcal{A}$; its Carathéodory extension to $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\mathcal{A})$ is the measure that we need. It remains to prove that if $B_1 \supset B_2 \supset \cdots$ are in $\mathcal{A}$ and $\cap_{n=1}^{\infty} B_n = \varnothing$, then $\lim_{n\to\infty}\mu(B_n) = 0$; see the proof of Lemma 3.15. Now we modify the proof of Lemma 3.15 slightly; after all, Lemma 3.15 is the present problem with $F(x) := x$ for $0 \leqslant x \leqslant 1$. In fact, all we have to do is to replace (3.6) with

$$F(\alpha_j^n) \leqslant F(a_j^n) + \frac{\epsilon}{2^{n+2}k_n}. \tag{3.6$'$}$$

[We can do this because F is right continuous.] Then, the remainder of the proof of Lemma 3.15 goes through *verbatim*.