Chapter 6

Independence

6.10. (i)⇒(ii) If \( X_1 \in L^p(P) \) then \( \sum_{n=1}^{\infty} P(|X_n| > \varepsilon n^{1/p}) = \sum_{n=1}^{\infty} P(|X_1| > \varepsilon n) \leq \frac{E(|X_1|)}{\varepsilon}. \) So by the Borel–Cantelli lemma, with probability one, \( |X_n| \leq \varepsilon n^{1/p} \) for all \( n \) large. This proves that \( |X_n|/n^{1/p} \to 0 \) a.s.

(ii)⇒(iii) This follows from the following real-variable Fact. If \( a_n/n^p \to 0 \) for some \( p > 0 \), then \( \max_{1 \leq j \leq n} a_j \to 0 \).

Proof: If not, then \( \max_{1 \leq j \leq n} a_j > \varepsilon n^p \) infinitely often. But there exists \( n_0 \) such that for all \( n \geq n_0 \) \( |a_n| \leq (\varepsilon/2)n^p \). So \( \max_{1 \leq n \leq n_0} a_j > \varepsilon n^p \) for infinitely-many \( n \)'s, which is patently nonsense.

(iii)⇒(i) Because (iii) implies (ii), it suffices to prove that (ii) implies (i). But this too is Borel–Cantelli (as in i⇒ii).

6.11. By the Kolmogorov maximal inequality, \( \lambda^2 P(\max_{1 \leq j \leq n} |S_j| \geq \lambda) \leq ES_n^2 \leq \sum_{i=1}^{\infty} EX_i^2 = A \) (Fubini–Tonelli). Therefore,

\[
P \left\{ \sup_{n \leq j \leq n+m} |S_j - S_n| \geq \lambda \right\} \leq \frac{\sum_{i=n}^{n+m} E|X_i^2|}{\lambda^2} \quad \forall \lambda > 0.
\]

Continuity properties of \( P \) imply that

\[
P \left\{ \sup_{j \geq n} |S_j - S_n| \geq \lambda \right\} \leq \frac{\sum_{i=n}^{\infty} E|X_i^2|}{\lambda^2} \quad \forall \lambda > 0.
\]

We can find \( n_k \uparrow \infty \) so fast that \( \sum_{i \geq n_k} E|X_i^2| \leq e^{-k} \). By Borel-Cantelli then,

\[
P \left\{ \sup_{j \geq n_k} |S_j - S_{n_k}| \geq k^{-2} \right\} \leq k^4 e^{-k},
\]

which is summable in \( k \). Therefore, the Borel–Cantelli lemma ensures that a.s. for all but a finite number of \( k \)'s, \( \sup_{j \geq n_k} |S_j - S_{n_k}| \leq k^{-2} \). In
particular, \( \sum_k \sup_{j \geq n_k} |S_j - S_{n_k}| < \infty \) a.s. This implies that \( \{S_n\}_{n=1}^{\infty} \) is a.s. a Cauchy sequence (why?).

6.13. Fix some \( \nu > 0 \). Clearly, \( \liminf_n (S_n/\nu) \geq \liminf_n (S_n^{\nu}/\nu) \), where \( S_n^{\nu} := \sum_{i=1}^{n} X_i^{\nu} \), and \( X_i^{\nu} := X_i 1{X_i \leq \nu} \). By the law of large numbers, \( \liminf_n (S_n^{\nu}/\nu) = \lim_n (S_n^{\nu}/\nu) = E[X_1; X_1 \leq \nu] \). Let \( \nu \uparrow \infty \) to finish.

6.14. By Chebyshev’s inequality, for all \( \varepsilon > 0 \),

\[
P \left\{ \left| \frac{S_n - n\mu}{\varepsilon} \right| > \frac{\varepsilon}{\ln(n)} \right\} \leq \frac{\Var S_n}{\varepsilon^2 \ln^2(n)}
\]

Thus there exists \( C > 0 \) and \( n_0 \geq 1 \) such that for all \( n \geq n_0 \), the preceding probability is \( \leq C \varepsilon^{-2} n^{-\delta} \). Replace \( n \) by \( n^k \), where \( k > 1/\delta \) is a fixed integer, and then use Borel-Cantelli, to find that \( S_{n^{k}}/n^k \to \mu \). If \( n^k \leq m \leq (n+1)^k \) then

\[
\frac{S_m}{m} \leq \frac{S_{(n+1)^k} - S_{n^k}}{n^k} = \frac{S_{(n+1)^k} - (n+1)^k}{n^k} \to \mu.
\]

Similarly, \( S_m/m \geq \mu + o(1) \) a.s. Thus, \( S_m/m \sim \mu \) a.s. Now suppose the \( X_i \)'s are identically distributed as well as uncorrelated. Then, \( \Var S_n = n \Var X_1 = o(n^{2-\delta}) \) for some \( \delta \in (0, 2) \). Thus, \( S_n \to E X \) a.s.

6.29. Solution 1. We will need the following Lemma: As \( n \to \infty \), \( \sum_{i=1}^{n} (1/i) \sim \ln(n) \).

\textbf{Proof } This follows from the integral test of calculus, because \( \int_{1}^{n} dx/x \leq \sum_{i=1}^{n} (1/i) \leq 1 + \int_{1}^{n} dx/x \).

Now let us first assume that \( E[X_1] = 0 \). Define \( X'_i = X_i 1{|X_i| \leq i} \). Also define \( S_n = \sum_{i=1}^{n} (X_i/i) \) and \( S'_n = \sum_{i=1}^{n} (X'_i/i) \). By the Kolmogorov maximal inequality,

\[
P \left\{ \max_{1 \leq k \leq n} |S'_k - E[S'_k]| \geq \varepsilon \ln n \right\} \leq \frac{\Var S'_n}{\varepsilon^2 (\ln n)^2} \leq \frac{E[S'_n]^2}{\varepsilon^2 \ln^2(n)} = \frac{1}{\varepsilon^2 \ln^2(n)} \sum_{i=1}^{n} \frac{E[X'_i] 1{|X| \leq i}}{i^2}.
\]
Therefore,
\[ \sum_{n=1}^{\infty} P \left\{ \max_{1 \leq k \leq 2^n} |S'_k - E[S'_k]| \geq \epsilon \ln \left(2^{2^n}\right) \right\} \leq A \sum_{n=1}^{\infty} \frac{1}{4^n} \sum_{i=1}^{2^n} \frac{E[X_i^2; |X_i| \leq i]}{i^2} \]
\[ \leq A \sum_{i=1}^{\infty} \frac{E[X_i^2; |X_i| \leq i]}{i^2} \sum_{n \geq \log_2(i)} \frac{1}{4^n} \]
\[ \leq B \sum_{i=1}^{\infty} \frac{E[X_i^2; |X_i| \leq i]}{i^2} \]
\[ = BE \left[ X_i^2 \sum_{i \geq |X_i|} \frac{1}{i^2} \right] \leq \text{CE}_1. \]

By the Borel–Cantelli lemma, a.s., for all \( n \) large,
\[ \max_{1 \leq k \leq 2^n} |S'_k - E[S'_k]| \leq \epsilon \ln \left(2^{2^n}\right). \]

Any \( m \) is between some \( 2^n \) and \( 2^{n+1} \). Therefore,
\[ |S'_m - E[S'_m]| \leq \max_{1 \leq k \leq 2^{n+1}} |S'_k - E[S'_k]| \leq \epsilon \ln \left(2^{2^{n+1}}\right) = 2\epsilon \ln(2) \cdot 2^n \leq 2\epsilon \ln(2) \ln m. \]

This proves that a.s., \( |S'_m - E[S'_m]|/\ln(m) \to 0 \). But \( \sum_k P(S_k \neq S_k') = \sum_k P(|X_k| > k) < \infty \) because \( \|X_k\| < \infty \). So almost surely, \( S_k = S_k' \) for all \( k \) large. It suffices to prove that \( |E[S'_m]|/\ln m \to 0 \). But
\[ |E[S'_m]| = \sum_{i=1}^{m} \frac{E[|X_i|; |X_i| \leq i]}{i} = \sum_{i=1}^{m} \frac{E[|X_i|; |X_i| > i]}{i}, \]
since \( EX_1 = 0 \). Thus,
\[ |E[S'_m]| \leq \sum_{i=1}^{m} \frac{E[|X_i|; |X_i| > i]}{i}. \]

For all \( \eta > 0 \), there exists \( i_0 \) such that for all \( i \geq i_0 \), \( E[|X_i|; |X_i| > i] \leq \eta \).
Therefore, \( \sum_{i=i_0}^{m} \eta \leq \eta \sum_{i=1}^{m} (1/i) \sim \eta \ln m. \) But \( \sum_{i=1}^{kn} (1/i) \leq \eta \ln X_1. \)
Therefore, \( \lim_{m \to \infty} |E[S'_m]|/\ln m \leq \eta \) for all \( \eta \), whence the result.

If \( \mu = EX_1 \neq 0 \), then consider instead \( X'_i := X_i - \mu \). The preceding proves that \( \ln^{-1}(n) \sum_{i=1}^{n} X'_i/i \to 0 \) a.s. Equivalently, \( \ln^{-1}(n) \sum_{i=1}^{n} (X_i/i - \mu \sum_{i=1}^{n} (1/i)) \to 0 \). Because \( \sum_{i=1}^{n} (1/i) \sim \ln(n) \), we obtain the result in general.

**Solution 2.** We may start with the following real-variable lemma. **Lemma:**
If \( a_n \to \mu \) and \( b_n \geq 0 \) satisfy \( \sum_{i=1}^{n} b_i \to \infty \), then \( \sum_{i=1}^{n} a_i b_i \sim \mu \sum_{i=1}^{n} b_i. \)
Proof. Fix $\varepsilon > 0$, and find $n_0$ so large that $|a_i - \mu| \leq \mu + \varepsilon$ for all $i \geq n_0$. Then,

$$\sum_{i=1}^{n} a_ib_i \sim \sum_{i=n_0}^{n} a_ib_i = (\mu \pm \varepsilon) \sum_{i=n_0}^{n} b_i \sim (\mu \pm \varepsilon) \sum_{i=1}^{n} b_i,$$

notation being clear.  

Now let $S_0 = 0$, and $S_n = \sum_{j=1}^{n} S_j (n \geq 1)$, so that

$$\sum_{i=1}^{n} \frac{X_i}{i} = \sum_{i=1}^{n} \left( S_i - S_{i-1} \right) \frac{1}{i} = \sum_{i=1}^{n} S_i \frac{1}{i} - \sum_{i=1}^{n} S_{i-1} \frac{1}{i}$$

$$= \sum_{i=1}^{n} S_i \frac{1}{i} - \sum_{i=2}^{n-1} S_i \frac{1}{i+1} = S_1 + \sum_{i=2}^{n-1} S_i \left( \frac{1}{i} - \frac{1}{i+1} \right) - S_n \frac{1}{n+1}.$$ 

By the strong law, $S_n/(n+1) \to \mu$ a.s. Therefore,

$$\frac{1}{\ln n} \sum_{i=1}^{n} \frac{X_i}{i} \sim \frac{1}{\ln n} \sum_{i=2}^{n-1} \frac{S_i}{i(1+1)} \sim \frac{1}{\ln n} \sum_{i=2}^{n-1} \frac{\mu}{i+1} \to \mu \quad \text{a.s.}$$

6.32. Let $S_n$ denote the number of $X_i$'s ($1 \leq i \leq n$) that are equal to one. Thus, $S_n = \text{Bin}(n, 1/2)$. By the De Moivre–Laplace central limit theorem, for all $\lambda > 0$,

$$\lim_{n \to \infty} P \left\{ S_n \geq \frac{n}{2} + \lambda \sqrt{n} \right\} = P\{N(0, 1) \geq \lambda\} > 0.$$ 

Thus, by the Kolmogorov 0–1 law,

$$\limsup_{n \to \infty} \frac{S_n - (n/2)}{\sqrt{n}} = \infty \quad \text{a.s.}$$

Because $\sum_{i=1}^{n} X_i = 2S_n - n$, it follows that $\limsup_{n \to \infty} \sum_{i=1}^{n} X_i/\sqrt{n} = \infty$ a.s. By symmetry, $\liminf_{n \to \infty} \sum_{i=1}^{n} X_i/\sqrt{n} = -\infty$ a.s. Thus,

$$\limsup_{n \to \infty} \sum_{i=1}^{n} X_i = -\liminf_{n \to \infty} \sum_{i=1}^{n} X_i = \infty \quad \text{a.s.}$$

6.33.

1. First of all we note that if $n \geq 1$ and $t \geq 0$, then for any arbitrary $\lambda > 0$,

$$P(S_n \geq t) = P \left\{ e^{\lambda S_n} \geq e^{\lambda t} \right\} \leq e^{-\lambda t} E \left[ e^{\lambda S_n} \right],$$

owing to Chebyshev’s inequality. But $E \exp(\lambda S_n) = \prod_{j=1}^{n} E \exp(\lambda X_j)$, because the expectation of a product of independent random variables is the product of the expectations. Problem 4.30 of the text implies that $E \exp(\lambda X_j) \leq \exp(\lambda^2 c_j^2/2)$. Therefore,

$$E \left[ e^{\lambda S_n} \right] \leq e^{\lambda^2 s_n^2/2}.$$
This proves that
\[ P\{|S_n| \geq t\} \leq e^{-\lambda t + \lambda^2 s_n^2/2}. \]
The left-hand side is independent of \( \lambda \). So we can minimize the right-hand side over all \( \lambda > 0 \); the minimum occurs at \( \lambda = t/s_n \), and yields \( P\{|S_n| \geq t\} \leq \exp(-t/s_n^2) \). The result follows from replacing \( S_n \) by \( -S_n \).

2. Let \( I_1, \ldots, I_n \) be i.i.d. \( \text{Bin}(1, p) \) random variables. Then \( B \) has the same distribution as \( I_1 + \cdots + I_n \). Let \( X_i := I_i - p \). Then the \( X_i \)'s are i.i.d., have mean zero, and are bounded between \(-1\) and \(1\). Therefore, we can apply the first part to \( S_n := X_1 + \cdots + X_n \) to find that
\[ s_n^2 := n \quad \text{and} \quad P\{|S_n| \geq t\} \leq 2e^{-t/(2n)} \quad \text{for all } t \geq 0. \]
Replace \( t \) by \( t\sqrt{n} \) and observe that \( S_n \) has the same distribution as \( B - np \) to finish.

3. Note that, in (6.48), \( |T_2| \leq KP(A^c) \), and \( A^c = \{|S_n - np| > n\delta\} \). Apply the previous part with \( t := \delta\sqrt{n} \) to finish.

4. Similar to part 3.
Chapter 7

The Central Limit Theorem

7.1. Since $C^\infty_c(R^k) \subset C_b(R^k)$, it suffices to prove that if

$$\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu \quad \text{for all } f \in C^\infty_c(R^k),$$

then $\mu_n \Rightarrow \mu$.

Claim. For every $f \in C_c(R^k)$ and $\varepsilon \in (0, 1)$, there exists $g_\varepsilon \in C^\infty_c(R^k)$ such that $\sup_x |f(x) - g_\varepsilon(x)| \leq \varepsilon$.

This Claim does the job. Indeed, the Claim implies that

$$\left| \int f \, d\mu_n - \int g_\varepsilon \, d\mu_n \right| \leq \varepsilon \mu(R^k) = \varepsilon, \text{ and}$$

$$\left| \int f \, d\mu - \int g_\varepsilon \, d\mu \right| \leq \varepsilon.$$

Consequently,

$$\limsup_{n \to \infty} \left| \int f \, d\mu_n - \int f \, d\mu \right| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ and $f \in C_c(R^k)$ are arbitrary, the result follows.

Now by Fejer’s theorem, if $f \in C_c(R^k)$ then for all $\varepsilon > 0$ there exists $h_\varepsilon \in C^\infty_c(R^k)$ such that $\sup_x |f(x) - h_\varepsilon(x)| \leq \varepsilon/2$. Suppose $f(x) = 0$ when $\|x\| > p$. Then it suffices to prove that there exists $g_\varepsilon \in C^\infty_c(R^k)$ such that $\sup_{\|x\| \leq p} |g_\varepsilon(x) - h_\varepsilon(x)| \leq \varepsilon$. But this is easy. For instance, let

$$\psi(x) := \exp \left( -\frac{1}{(p^2 - \|x\|)^{+}} \right).$$

Then, $\psi \in C^\infty_c(R^k)$, $0 \leq \psi(x) \leq 1$ for all $x$, and $\psi(x) \geq \exp(-1/p(p-1)) \geq 1 - \varepsilon$ when $\|x\| < p$, if $p = p(\varepsilon)$ is chosen sufficiently large. Then $g_\varepsilon(x) := \psi(x)h_\varepsilon(x)$ satisfies

$$(1 - \varepsilon)h_\varepsilon(x) \leq g_\varepsilon(x) \leq h_\varepsilon(x) \quad \text{when } \|x\| < p.$$
Therefore,

\[
\sup_{\|x\| \leq p} |g_e(x) - h_e(x)| \leq \max \left( \sup_x |h_e(x)| , 1 \right) \leq K \epsilon,
\]

where \(K := \sup_x |f(x)| + \epsilon \leq \sup_x |f(x)| + 1\).