8.38. Almost surely,

$$\mathrm{E}[X_{n+1} \mid \mathcal{F}_n] = \Big\{\lambda + (1-\lambda)X_n\Big\}X_n + (1-\lambda)X_n(1-X_n) = X_n.$$

Therefore, $\{X_n\}_{n=1}^\infty$ is a bounded martingale, and hence $X_\infty := \lim_{n\to\infty} X_n$ exists a.s. and in $L^1(\mathrm{P})$. By the dominated convergence theorem convergence holds in $L^p(\mathrm{P})$ too, where $p \geq 1$. Define

$$\theta := \mathrm{E}X_1.$$

Then, $\theta \in [0\,,1]$, and $\theta = \mathrm{E}X_n$ for all n by the martingale property, and $\theta = \mathrm{E}X_\infty$ by L^1-convergence. By the towering property of conditional expectations,

$$\theta = \mathrm{P}\left\{X_{n+1} = \lambda + (1-\lambda)X_n\right\}.$$

Therefore,

$$\theta = \mathrm{P}\left\{X_\infty = \lambda + (1-\lambda)X_\infty\right\} = \mathrm{P}\left\{X_\infty = 1\right\}$$

(Hint: Fatou's lemma, and the reversed Fatou's lemma for bounded random variables.) Similarly,

$$1 - \theta = \mathrm{P}\left\{X_\infty = -\lambda X_\infty\right\} = \mathrm{P}\left\{X_\infty = 0\right\}.$$