8.36. We will need the following variant of the optional stopping theorem. **Theorem:** *Suppose* $\{X_n\}_{n=1}^{\infty}$ *is a non-negative supermartingale with respect to a filtration* $\mathcal{F} := \{\mathcal{F}_n\}_{n=1}^{\infty}$, *and* T *is an* $\mathcal{F}$*-stopping time. Then,*

$$\mathrm{E}\left[X_n \mid \mathcal{F}_T\right] \le X_T \quad \textit{a.s. on} \quad \{T < n\}.$$

Proof: Let $d_0 := 0$ and $\mathcal{F}_0 := \{\varnothing, \Omega\}$, then define $d_{n+1} := X_{n+1} - X_n$ for all $n \ge 0$. Then, $X_n = \sum_{j=1}^n d_j$, and $\mathrm{E}[d_{j+1} \,|\, \mathcal{F}_j] \le 0$. Almost surely,

$$X_n \mathbf{1}_{\{T<n\}} = X_T \mathbf{1}_{\{T<n\}} + \sum_{j=1}^n d_j \mathbf{1}_{\{T<j\}}$$

Therefore, for all $A \in \mathcal{F}_T$,

$$\mathrm{E}\left[X_n \mathbf{1}_{\{T<n\}} \; ; \; A\right] = \mathrm{E}\left[X_T \mathbf{1}_{\{T<n\}} \; ; \; A\right] + \sum_{j=1}^n \mathrm{E}\left[d_j \; ; \; A \cap \{T < j\}\right] \le \mathrm{E}\left[X_T \mathbf{1}_{\{T<n\}} \; ; \; A\right],$$

because $A \cap \{T < j\} \in \mathcal{F}_{j-1}$. This proves the theorem. □

Now we return to the problem at hand. By the martingale convergence theorem $X_\infty := \lim_{n\to\infty} X_n$ exists and is finite a.s. Let $T := \inf\{n \ge 1 : X_n = 0\}$, and note that we we are told that T is finite a.s. Apply Fatou's lemma to find that

$$\mathrm{E}X_\infty \le \liminf_{n\to\infty} \mathrm{E}\left[X_n; T < n\right] = \liminf_{n\to\infty} \mathrm{E}\left[\mathrm{E}\left(X_n \mid \mathcal{F}_T\right); T < n\right].$$

By the stated theorem, this is at most $\liminf_n \mathrm{E}[X_T; T < n] \le \mathrm{E}X_T = 0$. Since $X_\infty \ge 0$ a.s., X_∞ must be zero a.s. In fact, a small modification of this proof shows that with probability one, $X_{T+n} = 0$ for all $n \ge 1$.