8.26. Define $\mathcal{F}_n$ to the collection all sets $X^{-1}(I)$, where I is a dyadic interval. Thus, $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is a filtration, and $\mathrm{E}[Y \mid X_n] = \mathrm{E}[Y \mid \mathcal{F}_n]$. Because it is bounded in $L^1(\mathrm{P})$, $M_n := \mathrm{E}[Y \mid X_n]$ has an a.s. limit M_∞. Evidently, M_∞ is measurable with respect to $\mathcal{F}_\infty := \vee_{n=1}^{\infty} \mathcal{F}_n := \sigma(X)$.

First suppose in addition Y is bounded, so that $M_n \to M_\infty$ in $L^1(\mathrm{P})$ as well. Because conditional expectations are contractions on $L^1(\mathrm{P})$, this shows that $M_n = \mathrm{E}[M_{n+k} \mid \mathcal{F}_n] \to \mathrm{E}[M_\infty \mid \mathcal{F}_n]$ in $L^1(\mathrm{P})$, as $k \to \infty$. Thus, $\mathrm{E}(M_\infty \mid \mathcal{F}_n) = M_n$ a.s. for all $n \geq 1$. Thus, for all $A \in \mathcal{F}_\infty$, $\mathrm{E}[M_\infty; A] = \mathrm{E}[M_n; A] = \mathrm{E}[Y; A]$. This proves that $\mathrm{E}[M_\infty; A] = \mathrm{E}[Y; A]$ for all $A \in \mathcal{F}_\infty$. Therefore, $M_\infty = \mathrm{E}[Y \mid \mathcal{F}_\infty] = \mathrm{E}[Y \mid X]$.

Next suppose $Y \geq 0$. What we just proved shows that $\mathrm{E}[Y \wedge k \mid X_n] \to \mathrm{E}[Y \wedge k \mid X]$ a.s. as $n \to \infty$. By Doob's inequality, for all $\lambda > 0$,

$$\mathrm{P}\left\{\sup_{n \geq 1} \Big|\mathrm{E}[Y \mid X_n] - \mathrm{E}[Y \wedge k \mid X_n]\Big| > \lambda\right\} \leq \frac{1}{\lambda}\mathrm{E}[Y;\, Y > k].$$

In particular,

$$\mathrm{P}\left\{\limsup_{n \to \infty} \Big|\mathrm{E}[Y \mid X_n] - \mathrm{E}[Y \mid X]\Big| > \lambda\right\} \leq \frac{1}{\lambda}\mathrm{E}[Y;\, Y > k].$$

Let $k \to \infty$ to find that when Y is nonnegative, $\lim_{n\to\infty} \mathrm{E}[Y \mid X_n] = \mathrm{E}[Y \mid X]$. The general result follows from this and writing $Y = Y^+ - Y^-$.

8.27. Without loss of too much generality we consider the case where X is one-dimensional. Let X_n be as in **8.26**, and note that $\mathrm{E}[Y \mid X_n]$ is a nonrandom Borel function of X_n. The result follows from **8.26**.