8.22. We integrate by parts:

$$\begin{aligned}\|\xi\|_p^p &= p\int_0^\infty \lambda^{p-1}\mathrm{P}\{\xi > \lambda\}\,d\lambda \le p\int_0^\infty \lambda^{p-2}\mathrm{E}\left[\zeta;\ \xi > \lambda\right]\,d\lambda \\ &= p\mathrm{E}\left[\zeta\int_0^\xi \lambda^{p-2}\,d\lambda\right] = \frac{p}{p-1}\mathrm{E}\left[\zeta\xi^{p-1}\right],\end{aligned}$$

by Fubini–Tonelli. Let $a > 1$ be fixed, and define b by $a^{-1} + b^{-1} = 1$. Then, Hölder's inequality shows us that

$$\|\xi\|_p^p \le \frac{p}{p-1}\left(\mathrm{E}[\zeta^a]\right)^{1/a}\left(\mathrm{E}\left[\xi^{b(p-1)}\right]\right)^{1/b}.$$

Apply this with $a := p$, so that $b = p/(p-1)$, and we get

$$\|\xi\|_p^p \le \frac{p}{p-1}\|\zeta\|_p^p \cdot \|\xi\|_p^{(1/p)-1}.$$

If $\|\xi\|_p = 0$ then there is nothing to prove. Else, $\|\xi\|_p > 0$, and we can divide the preceding by $\|\xi\|_p$ and solve to finish the first portion.

Apply the said result with $\xi := \max_{1\le i\le n} X_i$ and $\zeta := X_n$ to obtain Doob's maximal inequality (8.70). Finally, apply this to the proof of Theorem 8.20 to finish.