**4.31.** There are two cases to consider: (1) $0 \le \alpha \le 1$; and (2) $\alpha > 1$.

1. Suppose $\alpha \in [0,1]$. By Taylor expansions there exists $c > 0$ such that $\exp(x/n) \le 1 + c(x/n)$ for all $x \in [0,n]$. That is,

$$\sup_{n \ge 1} \frac{\exp(x/n) - 1}{x/n} \le c \quad {}^{\forall} x \in [0,n].$$

   Also, for each $x$ fixed,

$$\lim_{n\to\infty} \frac{\exp(x/n) - 1}{x/n} = 1.$$

   Therefore, by the dominated convergence theorem, since $0 \le \alpha \le 1$,

$$\lim_{n\to\infty} n \int_0^{n^\alpha} \frac{\exp(x/n) - 1}{x + x^3}\, dx = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

2. Now suppose $\alpha > 1$, and observe that there exists $c > 0$ such that for all $n$ large enough,

$$\int_0^{n^\alpha} \frac{\exp(x/n) - 1}{x + x^3}\, dx \ge \int_{n^\alpha/2}^{n^\alpha} \frac{\exp\left(n^{\alpha-1}\right) - 1}{2x^3}\, dx \ge cn^{-2\alpha} \exp\left(n^{\alpha-1}\right)$$

   Therefore,

$$\lim_{n\to\infty} n \int_0^{n^\alpha} \frac{\exp(x/n) - 1}{x + x^3}\, dx = \infty.$$