4.20. For all $r > 0$, $1 - r \le e^{-r}$. Therefore,

$$\left(1 - \frac{x^2}{2n}\right)^n \le e^{-x^2/2},$$

for all n, and the left-hand side also converges to the right-hand side as $n \to \infty$. Therefore, by the dominated convergence theorem the integral of the problem converges to $\int_{-\infty}^{\infty} \exp(-x^2/2)\, dx$. It remains to compute the integral. Here is one way:

$$\int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{x^2}{2n}\right)^n dx = 2\sqrt{n} \int_0^1 \left(1 - \frac{y^2}{2}\right)^n dy = \sqrt{2n} \int_{1/2}^1 x^n (1-x)^{-1/2}\, dx.$$

Evidently,

$$\sqrt{n} \int_0^{1/2} x^n (1-x)^{-1/2}\, dx \le \sqrt{n} 2^{-n} \int_0^{1/2} (1-x)^{-1/2}\, dx \to 0.$$

Thus,

$$\int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{x^2}{2n}\right)^n dx \sim \sqrt{2n} \int_0^1 x^n (1-x)^{-1/2}\, dx = \sqrt{2n} \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \sim \sqrt{2n\pi} \frac{n^{n+(1/2)} e^{-n} \beta}{(n+\frac{1}{2})^{n+1} e^{-n-(1/2)} \beta} \to \sqrt{2\pi}.$$