4.1. There are two things to prove here: (i) $\sigma(X)$ is σ-algebra; and (ii) it is the smallest one with respect to which X is measurable.

As regards (i), we check that $\varnothing \in \sigma(X)$ because $X^{-1}(\varnothing) = \varnothing$. Also if $A \in \sigma(X)$ then $A = X^{-1}(B)$ for some $B \in \mathscr{A}$. But then $A^c = (X^{-1}(B))^c$, which is in $\sigma(X)$. Finally, suppose $A_1, A_2, \ldots$ are all in $\sigma(X)$. Then we can find $B_1, B_2, \ldots$ such that $A_i = X^{-1}(B_i)$. Evidently, $\cup_{i=1}^{\infty} X^{-1}(B_i) = X^{-1}(\cup_{i=1}^{\infty} B_i)$. Because $\cup_{i=1}^{\infty} B_i \in \mathscr{A}$ (the latter is after all a σ-algebra), it follows that $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} X^{-1}(B_i) \in \sigma(X)$. We have proved that $\sigma(X)$ is a σ-algebra.

Note that X is measurable with respect to a σ-algebra $\mathscr{G}$ iff $X^{-1}(B) \in \mathscr{G}$ for all $B \in \mathscr{A}$. Therefore, a priori, X is measurable with respect to $\sigma(X)$, and any other $\mathscr{G}$ must contain $\sigma(X)$.