MORE APPLICATIONS OF MARTINGALE THEORY

DAVAR KHOSHNEVISAN

1. Patterns

Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with $\mathrm{P}\{X_1 = 1\} = p$ and $\mathrm{P}\{X_1 = 0\} = q$, where $q = 1 - p$ and $0 < p < 1$. It is possible to use the Kolmogorov zero-one and deduce that the infinite sequence $X_1, X_2, \ldots$ will a.s. contain a zero, say. In fact, any pre-described finite pattern of zeroes and ones will appear infinitely often in the sequence $\{X_1, X_2, \ldots\}$ with probability one. Let N denote the first k such that the sequence $\{X_1, \ldots, X_k\}$ contains a pre-described, non-random pattern. Then, we wish to know $\mathrm{E}[N]$.

The simplest patterns are "0" and "1." So let N denote the smallest k such that $\{X_1, \ldots, X_k\}$ contains a "0." It is not hard to convince yourself that $\mathrm{E}[N] = 1/q$ because N has the following geometric distribution:

$$\mathrm{P}\{N = j\} = p^{j-1} q \qquad j = 1, 2, \ldots. \tag{1.1}$$

This calculation uses too much of the structure of the pattern "0." Here is an alternative argument, due to Robert Li Shuo-Yen (1980),[1] which is more robust:

Consider the process

$$Y_n := \frac{1}{q}\mathbf{1}_{\{X_1=0\}} + \frac{1}{q}\mathbf{1}_{\{X_2=0\}} + \cdots + \frac{1}{q}\mathbf{1}_{\{X_n=0\}}. \tag{1.2}$$

Define $\mathscr{F}_n$ to be the σ-algebra defined by $\{X_i\}_{i=1}^n$ for every $n \geq 1$. Then, for all $n \geq 1$,

$$\mathrm{E}[Y_{n+1} \,|\, \mathscr{F}_n] = Y_n + \frac{1}{q}\mathrm{P}(X_{n+1} = 0 \,|\, \mathscr{F}_n) = Y_n + 1. \tag{1.3}$$

Therefore, $\{Y_n - n\}_{n=1}^\infty$ is a mean-zero martingale (check!). By the optional stopping theorem,

$$\mathrm{E}\left[Y_{N\wedge n} - (N \wedge n)\right] = 0 \qquad {}^\forall n \geq 1. \tag{1.4}$$

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[1]A martingale approach to the study of occurance of sequence patterns in repeated experiments, *Ann. Probab.*, Vol. 8, No. 6, pp. 1171–1176 (1980)

Equivalently, $\mathrm{E}[N \wedge n] = \mathrm{E}[Y_{N\wedge n}]$. But $N < \infty$ a.s., and both $\{N \wedge n\}_{n=1}^\infty$ and $\{Y_{N\wedge n}\}_{n=1}^\infty$ are increasing. Therefore, we can apply the monotone convergence theorem to deduce that

$$\mathrm{E}[Y_N] = \mathrm{E}[N]. \tag{1.5}$$

On the other hand, $Y_N = (1/q)$ a.s. Therefore, $\mathrm{E}[N] = 1/q$, as we know already.

The advantage of the second proof is that it can be applied to other patterns. Suppose, for instance, the pattern is a sequence of ℓ ones, where ℓ is an integer greater than or equal to one. Consider

$$\begin{aligned} Z_{n,\ell} := {} & \frac{1}{p^\ell}\mathbf{1}_{\{X_1=1,\ldots,X_\ell=1\}} + \frac{1}{p^\ell}\mathbf{1}_{\{X_2=1,\ldots,X_{\ell+1}=1\}} \\ & + \cdots + \frac{1}{p^\ell}\mathbf{1}_{\{X_{n-\ell}=1,\ldots,X_n=1\}} + \frac{1}{p^{\ell-1}}\mathbf{1}_{\{X_{n-\ell+1}=1,\ldots,X_n=1\}} \\ & + \frac{1}{p^{\ell-2}}\mathbf{1}_{\{X_{n-\ell+2}=1,\ldots,X_n=1\}} + \cdots + \frac{1}{p}\mathbf{1}_{\{X_n=1\}}. \end{aligned} \tag{1.6}$$

Then, you should check that $\{Z_{n,\ell} - n\}_{n=1}^\infty$ is a martingale. As before, we have $\mathrm{E}[Z_{N,\ell}] = \mathrm{E}[N]$, and now we note that $Z_{N,\ell} = (1/p^\ell) + (1/p^{\ell-1}) + \cdots + (1/p)$ a.s. Therefore,

$$\mathrm{E}[N] = \sum_{k=1}^{\ell} \frac{1}{p^k} = \frac{1}{q}\left(\frac{1}{p^\ell} - 1\right). \tag{1.7}$$

Therefore, set $\ell = 2$ to find that

$$\mathrm{E}[N] = \frac{1}{q}\left(\frac{1}{p^2} - 1\right) \quad \text{for the pattern “11.”} \tag{1.8}$$

Exercise 1. Prove that if the pattern is “01,” then $\mathrm{E}[N] = 1/(pq)$. Hint: First show that $\{W_n - n\}_{n=1}^\infty$ is a martingale, where

$$W_n := \frac{1}{pq}\mathbf{1}_{\{X_1=0,X_2=1\}} + \cdots + \frac{1}{pq}\mathbf{1}_{\{X_{n-1}=0,X_n=1\}} + \frac{1}{q}\mathbf{1}_{\{X_n=0\}}. \tag{1.9}$$

Which of the two patterns, “01” and “11,“ is more likely to come first? To answer this, we first note that $\{W_n - Z_{n,2}\}_{n=1}^\infty$ is a martingale, since it is the difference of two martingales.

Define T to be the smallest integer $k \geq 1$ such that the sequence $\{X_1, \ldots, X_k\}$ contains either “01” or “11.” Then, we argue as before and find that $\mathrm{E}[W_T - Z_{T,2}] = 0$. But $W_T - Z_{T,2} = (pq)^{-1}$ on

{"01" comes up first}, and $W_T - Z_{T,2} = (1/p) + (1/p^2) = (p+1)/p^2$ on {"11" comes up first}. Therefore,

$$\begin{aligned} 0 &= \mathrm{E}[W_T - Z_{T,2}] \\ &= \frac{1}{pq}\mathrm{P}\{\text{“01” comes up first}\} \\ &\qquad + \frac{p+1}{p^2}\left[1 - \mathrm{P}\{\text{“01” comes up first}\}\right]. \end{aligned} \tag{1.10}$$

Solve to find that

$$\mathrm{P}\{\text{“01” comes up before “11”}\} = \frac{p+1}{p+1+(p/q)}. \tag{1.11}$$

Thus,

$$\begin{aligned} &\mathrm{P}\{\text{“01” comes before “11”}\} < \mathrm{P}\{\text{“11” comes before “01”}\} \\ &\iff \quad \frac{p}{q} < p+1 \quad \iff \quad p > \frac{\sqrt{3}-1}{2}. \end{aligned} \tag{1.12}$$

Exercise 2. Find the probability that we see ℓ consecutive ones before k consecutive zeroes.

2. Quadratic Forms

Let $\{X_i\}_{i=1}^\infty$ be a sequence of i.i.d. random variables. For a given double array $\{a_{i,j}\}_{i,j=1}^\infty$ of real numbers, we wish to consider the "quadratic forms" process,

$$Q_n := \sum\sum_{1\le i,j\le n} a_{i,j} X_i X_j \qquad {}^\forall n \ge 1. \tag{2.1}$$

Define $a^*_{i,j} := (a_{i,j} + a_{j,i})/2$. A little thought shows that if we replace $a_{i,j}$ by $a^*_{i,j}$ in Q_n, then the value of Q_n remains the same. Therefore, we can assume that $a_{i,j} = a_{j,i}$, and suffer no loss in generality.

The quadratic form process $\{Q_n\}_{n=1}^\infty$ arises in many disciplines. For instance, in mathematical statistics, $\{Q_n\}_{n=1}^\infty$ belongs to an important family of processes called "U-statistics."

Here, we wish to prove the following theorem that is essentially due to Dale E. Varberg (1966)[2].

Theorem 2.1. *Suppose* $\mathrm{E}[X_1] = 0$, $\mathrm{E}[X_1^2] = 1$, *and* $\mathrm{E}[X_1^4] < \infty$. *If* $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{i,j}^2 < \infty$, *then* $\lim_{n\to\infty}(Q_n - \sum_{1\le i\le n} a_{i,i})$ *exists and is finite a.s.*

[2]Convergence of quadratic forms in independent random variables, *Ann. Math. Statist.*, Vol. 37, No. 3, pp. 567–576 (1966)

Proof. Let $A_n := \sum_{1\le i\le n} a_{i,i}$. Then, a direct computation reveals that $\{Q_n - A_n\}_{n=1}^\infty$ is a mean-zero martingale. We plan to prove that $\{Q_n - A_n\}_{n=1}^\infty$ is bounded in $L^2(\mathrm{P})$. Because boundedness in $L^2(\mathrm{P})$ implies boundedness in $L^1(\mathrm{P})$, the martingale convergence theorem does the rest.

Thanks to the symmetry of the $a_{i,j}$'s we can write

$$Q_n - A_n = 2U_n + V_n, \tag{2.2}$$

where

$$U_n := \sum\sum_{1\le i<j\le n} a_{i,j} X_i X_j \quad \text{and} \quad V_n := \sum_{1\le i\le n} a_{i,i}\left[X_i^2 - 1\right]. \tag{2.3}$$

Because $(x+y)^2 \le 2(x^2+y^2)$ for all $x, y \in \mathbf{R}$, it follows that

$$(Q_n - A_n)^2 \le 8U_n^2 + 2V_n^2. \tag{2.4}$$

Therefore,

$$\mathrm{E}\left[(Q_n - A_n)^2\right] \le 8\mathrm{E}[U_n^2] + 2\mathrm{E}[V_n^2]. \tag{2.5}$$

The second expectation is in fact the variance of the sum of n independent random variables. Therefore,

$$\mathrm{E}[V_n^2] = \sum_{1\le i\le n} a_{i,i}^2 \mathrm{Var}(X_1^2). \tag{2.6}$$

Therefore, $\sup_n \mathrm{E}[V_n^2] < \infty$. Similarly, we seek to prove that $\mathrm{E}[U_n^2]$ is bounded in n. In order to do this we compute directly to find that

$$\mathrm{E}[U_n^2] = \sum\sum_{1\le i<j\le n} a_{i,j}^2, \tag{2.7}$$

which is of course bounded. □

Exercise 3. Prove that $\mathrm{E}[U_n V_n] = 0$. Use it to show that

$$\mathrm{E}\left[(Q_n - A_n)^2\right] = 4\sum\sum_{1\le i<j\le n} a_{i,j}^2 + \mathrm{Var}(X_1^2)\sum_{1\le i\le n} a_{i,i}^2. \tag{2.8}$$

Department of Mathematics, The University of Utah, 155 S. 1400 E. Salt Lake City, UT 84112–0090, USA

E-mail address: davar@math.utah.edu

URL: http://www.math.utah.edu/~davar