Math 6020-1, Spring 2014
Partial Solutions to Assignment 2

7.8. Let $h_{i,i}$ be the ith diagonal entry of our hat matrix $\boldsymbol{H} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'$.

(a) $\boldsymbol{H}$ is symmetric and satisfies $\boldsymbol{H}^2 = \boldsymbol{H}$ [you should check the details].

(b) Note that $\boldsymbol{H}$ is positive definite, and so $\sum_{j=1}^n h_{j,j} = \mathrm{tr}(\boldsymbol{H}) = \sum_{j=1}^n \lambda_j$, where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $\boldsymbol{H}$. We have seen that, thanks to idempotence, the eigenvalues $\boldsymbol{H}$ are all ones and zeros; the number of ones is exactly the rank of $\boldsymbol{H}$, which is $r+1$. Therefore, $\sum_{j=1}^n \lambda_j = r+1$, which means that $\sum_{j=1}^n h_{j,j} = r+1$.

Let us arrange the e-pairs so that $\lambda_1 = \cdots = \lambda_{r+1} = 1$ and $\lambda_{r+2} = \cdots = \lambda_n = 0$. We can then write $\boldsymbol{H} = \boldsymbol{P}'\boldsymbol{D}\boldsymbol{P}$, where $\boldsymbol{P}$ is orthogonal and $\boldsymbol{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$; that is,

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{I}_{(r+1)\times(r+1)} & \boldsymbol{0}_{(r+1)\times(n-r-1)} \\ \boldsymbol{0}_{(n-r-1)\times(r+1)} & \boldsymbol{0}_{(n-r-1)\times(n-r-1)} \end{pmatrix}.$$

Then,

$$h_{j,j} = (\boldsymbol{P}'\boldsymbol{D}\boldsymbol{P})_{j,j} = \sum_{k=1}^n \sum_{l=1}^n P_{k,j} D_{k,l} P_{l,j} = \sum_{k=1}^{r+1} P_{k,j}^2.$$

Thus, $h_{j,j} \geq 0$ for all $j = 1, \ldots, n$. Also, $h_{j,j} = \sum_{k=1}^{r+1} P_{k,j}^2 \leq \sum_{k=1}^n P_{k,j}^2 = (\boldsymbol{P}'\boldsymbol{P})_{j,j} = 1$. Therefore, we see that

$$0 \leq h_{j,j} = \sum_{k=1}^{r+1} P_{k,j}^2 = 1 - \sum_{k=r+2}^{n} P_{k,j}^2 \leq 1 \qquad \text{for all } j = 1, \ldots, n.$$

Our task is to prove that the inequalities are strict [$0 < h_{j,j} < 1$]; equivalently, we need to verify that $0 < \sum_{k=1}^{r+1} P_{k,j}^2 < 1$ for all $j = 1, \ldots, n$. If not, then either $\sum_{k=1}^{r+1} P_{k,j}^2 = 0$ for some j, or $\sum_{k=r+2}^{n} P_{k,j}^2 = 0$ for some j. We show next that neither case can happen.

Suppose, to the contrary, that $h_{j,j} = \sum_{k=1}^{r+1} P_{k,j}^2 = 0$ for some j. If so, then the first $r+1$ items in the jth column of $\boldsymbol{P}$ are all zeros. Since the only non-zero terms in $\boldsymbol{D}$ are in the first $(r+1)\times(r+1)$ top quadrant, it would follows that the entire jth column of $\boldsymbol{DP}$ is all zeros. From this we could deduce that the jth column of $\boldsymbol{H} = \boldsymbol{P}'\boldsymbol{DP}$ is all zeros. Because $\boldsymbol{Z}'\boldsymbol{H} = \boldsymbol{Z}'$, it follows that the jth row of $\boldsymbol{Z}$ is all zeros, which yields a contradiction because $\boldsymbol{Z}$ is assumed to be full rank. This proves that $h_{j,j} > 0$ for all $1 \le j \le n$.

We may apply the preceding argument to the idempotent matrix $\boldsymbol{I} - \boldsymbol{H}$—in place of $\boldsymbol{H}$—in order to see that $1 - h_{j,j}$ also cannot be zero. Therefore, we have proved that $0 < h_{j,j} < 1$ for all $j \le n$, as desired.

(c) We are considering the linear model

$$y = \beta_0 + \beta_1 z + \epsilon.$$

Here,

$$\boldsymbol{Z} = \begin{pmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_n \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{Z}'\boldsymbol{Z} = \begin{pmatrix} n & \sum_{i=1}^n z_i \\ \sum_{i=1}^n z_i & \sum_{i=1}^n z_i^2 \end{pmatrix} = n \begin{pmatrix} 1 & \bar{z} \\ \bar{z} & \overline{z^2} \end{pmatrix}.$$

Since

$$\det(\boldsymbol{Z}'\boldsymbol{Z}) = n\left[\overline{z^2} - (\bar{z})^2\right] = n\left[\frac{1}{n}\sum_{i=1}^n z_i^2 - (\bar{z})^2\right] = \sum_{i=1}^n (z_i - \bar{z})^2,$$

it follows that

$$(\boldsymbol{Z}'\boldsymbol{Z})^{-1} = \frac{1}{\sum_{i=1}^n (z_i - \bar{z})^2} \begin{pmatrix} \overline{z^2} & -\bar{z} \\ -\bar{z} & 1 \end{pmatrix}$$

$$\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1} = \frac{1}{\sum_{i=1}^n (z_i - \bar{z})^2} \begin{pmatrix} \overline{z^2} - z_1\bar{z} & -\bar{z} + z_1 \\ \vdots & \vdots \\ \overline{z^2} - z_n\bar{z} & -\bar{z} + z_n \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{H} &= \frac{1}{\sum_{i=1}^n (z_i - \bar{z})^2} \begin{pmatrix} \overline{z^2} - z_1\bar{z} & -\bar{z} + z_1 \\ \vdots & \vdots \\ \overline{z^2} - z_n\bar{z} & -\bar{z} + z_n \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_n \end{pmatrix} \\ &= \frac{1}{\sum_{i=1}^n (z_i - \bar{z})^2} \begin{pmatrix} \overline{z^2} - 2z_1\bar{z} + z_1^2 & \bullet \\ \bullet & \overline{z^2} - 2z_n\bar{z} + z_n^2 \end{pmatrix}, \end{aligned}$$

where "$\bullet$" is a group of terms that I have not computed. The point is that we see from the preceding that

$$\begin{aligned} h_{j,j} &= \frac{1}{\sum_{i=1}^n (z_i - \bar{z})^2} \left[\overline{z^2} - 2z_j\bar{z} + z_j^2 \right] \\ &= \frac{1}{\sum_{i=1}^n (z_i - \bar{z})^2} \left[\overline{z^2} - (\bar{z})^2 + (z_j - \bar{z})^2 \right] \\ &= \frac{1}{\sum_{i=1}^n (z_i - \bar{z})^2} \left[\frac{1}{n} \sum_{k=1}^n (z_k - \bar{z})^2 + (z_j - \bar{z})^2 \right] \\ &= \frac{1}{n} + \frac{(z_j - \bar{z})^2}{\sum_{i=1}^n (z_i - \bar{z})^2}. \end{aligned}$$

7.9. Here,

$$\boldsymbol{Y} = \begin{pmatrix} 5 & -3 \\ 3 & -1 \\ 4 & -1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{Z} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that

$$\boldsymbol{H} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' = \begin{pmatrix} 0.6 & 0.4 & 0.2 & 0 & -0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ -0.2 & 0 & 0.2 & 0.4 & 0.6 \end{pmatrix}.$$

Therefore,

$$\widehat{\boldsymbol{Y}} = \boldsymbol{H}\boldsymbol{Y} = \begin{pmatrix} 4.8 & -0.3 \\ 3.9 & -1.5 \\ 3 & 0 \\ 2.1 & 1.5 \\ 1.2 & 3 \end{pmatrix}, \quad \widehat{\boldsymbol{\varepsilon}} = \begin{pmatrix} 0.2 & 0 \\ -0.9 & 0.5 \\ 1 & -1 \\ -0.1 & 0.5 \\ -0.2 & 0 \end{pmatrix}.$$

In particular,

$$\boldsymbol{Y}'\boldsymbol{Y} = \widehat{\boldsymbol{Y}}'\widehat{\boldsymbol{Y}} + \widehat{\boldsymbol{\varepsilon}}'\widehat{\boldsymbol{\varepsilon}} = \begin{pmatrix} 55 & -15 \\ 15 & 24 \end{pmatrix},$$

after direction computation.