Math 6010
Solutions to homework 1

1.2. (a) Notice that $X_1 - 2X_2 + X_3 = \boldsymbol{a}'\boldsymbol{X}$, where $\boldsymbol{a} := (1\,,-2\,,1)'$. Therefore, $\mathrm{Var}\,(X_1 - 2X_2 + X_3) = \boldsymbol{a}'\mathrm{Var}(\boldsymbol{X})\boldsymbol{a}$, which is

$$(1\,,-2\,,1)\begin{pmatrix}5 & 2 & 3\\ 2 & 3 & 0\\ 3 & 0 & 3\end{pmatrix}\begin{pmatrix}1\\ -2\\ 1\end{pmatrix} = (1\,,-2\,,1)\begin{pmatrix}4\\ -4\\ 6\end{pmatrix} = 18.$$

(b) Write $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$, where $\boldsymbol{A} := \begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$. Therefore, $\mathrm{Var}(\boldsymbol{Y}) = \boldsymbol{A}\mathrm{Var}(\boldsymbol{X})\boldsymbol{A}'$; that is,

$$\mathrm{Var}(\boldsymbol{Y}) = \begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}5 & 2 & 3\\ 2 & 3 & 0\\ 3 & 0 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}12 & 15\\ 15 & 21\end{pmatrix}.$$

1.4. The matrix of the quadratic form $(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2$ is

$$\boldsymbol{A} := \begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}.$$

According to Theorem 1.6 (p. 10), the variance of $(X_1 - X_2)^2 + (X_2 - X_3)^2 + (X_1 - X_3)^2 = \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ is

$$(\mu_4 - 3\mu_2^2)\boldsymbol{a}'\boldsymbol{a} + 2\mu_2^2\,\mathrm{tr}(\boldsymbol{A}^2),$$

because the means [i.e., the θ's in that theorem] are all zero. Here, $\boldsymbol{a} := (2\,,2\,,2)'$ is the diagonals-vector of $\boldsymbol{A}$,

$$\mu_2 = \mathrm{E}(X_i^2) = \int_{-1}^{1} \frac{1}{2}x^2\mathrm{d}x = \frac{1}{3},$$

and

$$\mu_4 = \mathrm{E}(X_i^4) = \int_{-1}^{1} \frac{1}{2}x^4\mathrm{d}x = \frac{1}{5}.$$

Therefore, $(\mu_4 - 3\mu_2^2)\boldsymbol{a}'\boldsymbol{a} = -\frac{24}{15} = -\frac{8}{5}$. Since $\boldsymbol{A}$ is symmetric,

$$\mathrm{tr}(\boldsymbol{A}^2) = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{i,j}A_{i,j} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{i,j}^2 = 18.$$

Therefore, the answer to this question is $-\frac{8}{5} + 4 = \frac{12}{5}$.

1.5. Observe that $\mathbf{X}'\mathbf{A}\mathbf{X}$ and $\mathbf{X}'\mathbf{B}\mathbf{X}$ are 1-dimensional random variables. Consequently,

$$\mathbf{X}'\mathbf{A}\mathbf{X}\cdot\mathbf{X}'\mathbf{B}\mathbf{X} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_iA_{i,j}X_j \cdot \sum_{k=1}^{n}\sum_{l=1}^{n} X_kB_{k,l}X_l = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} A_{i,j}B_{k,l}X_iX_jX_kX_l.$$

Take expectations to see that

$$\mathrm{E}\left[\mathbf{X}'\mathbf{A}\mathbf{X}\,,\mathbf{X}'\mathbf{B}\mathbf{X}\right] = \mathrm{E}\left(\mathbf{X}'\mathbf{A}\mathbf{X}\cdot\mathbf{X}'\mathbf{B}\mathbf{X}\right) = \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n A_{i,j}B_{k,l}\mathrm{E}\left(X_iX_jX_kX_l\right).$$

st Because the X_i's are independent and have mean and third moment zero, all of these terms are zero except in the following cases: (i) When $i=j=k=l$; (ii) $i=j\neq k=l$; (iii) $i=k\neq j=l$; (iv) $i=l\neq j=k$. In case (i), $\mathrm{E}(X_iX_jX_kX_l)=\mathrm{E}(X_1^4)=3\sigma^4$. In the remaining cases (i)–(iv), $\mathrm{E}(X_iX_jX_kX_l)=|\mathrm{E}(X_1^2)|^2=\sigma^4$. Therefore,

$$\mathrm{E}\left[\mathbf{X}'\mathbf{A}\mathbf{X}\,,\mathbf{X}'\mathbf{B}\mathbf{X}\right] = 3\sigma^4\sum_{i=1}^n A_{i,i}B_{i,i} + \sigma^4\sum_{i=1}^n\sum_{\substack{k=1\\k\neq i}}^n A_{i,i}B_{k,k} + \sigma^4\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n A_{i,j}B_{i,j} + \sigma^4\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n A_{i,j}B_{j,i},$$

considering each case separately in order from (i)–(iv). Because $\mathbf{A}$ and $\mathbf{B}$ are symmetric, the last two quantities are equal. This reduces the computation to the following:

$$\mathrm{E}\left[\mathbf{X}'\mathbf{A}\mathbf{X}\,,\mathbf{X}'\mathbf{B}\mathbf{X}\right] = 3\sigma^4\sum_{i=1}^n A_{i,i}B_{i,i} + \sigma^4\sum_{i=1}^n\sum_{\substack{k=1\\k\neq i}}^n A_{i,i}B_{k,k} + 2\sigma^4\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n A_{i,j}B_{i,j}.$$

Now,

$$\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n A_{i,j}B_{i,j} = \sum_{i=1}^n\sum_{j=1}^n A_{i,j}B_{i,j} - \sum_{i=1}^n A_{i,i}B_{i,i} = \mathrm{tr}(\mathbf{AB}) - \sum_{i=1}^n A_{i,i}B_{i,i}.$$

Consequently,

$$\begin{aligned}\mathrm{E}\left[\mathbf{X}'\mathbf{A}\mathbf{X}\,,\mathbf{X}'\mathbf{B}\mathbf{X}\right] &= \sigma^4\sum_{i=1}^n A_{i,i}B_{i,i} + \sigma^4\sum_{i=1}^n\sum_{\substack{k=1\\k\neq i}}^n A_{i,i}B_{k,k} + 2\sigma^4\mathrm{tr}(\mathbf{AB})\\ &= \sigma^4\sum_{i=1}^n\sum_{k=1}^n A_{i,i}B_{k,k} + 2\sigma^4\mathrm{tr}(\mathbf{AB})\\ &= \sigma^4\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) + 2\sigma^4\mathrm{tr}(\mathbf{AB})\\ &= \mathrm{E}\left(\mathbf{X}'\mathbf{A}\mathbf{X}\right)\mathrm{E}\left(\mathbf{X}'\mathbf{B}\mathbf{X}\right) + 2\sigma^4\mathrm{tr}(\mathbf{AB}),\end{aligned}$$

thanks to Theorem 1.5. This shows readily that

$$\mathrm{Cov}\left[\mathbf{X}'\mathbf{A}\mathbf{X}\,,\mathbf{X}'\mathbf{B}\mathbf{X}\right] = 2\sigma^4\mathrm{tr}(\mathbf{AB}).$$