Math 6010
Partial solutions to homework 5

1. We have a linear model with $p = 2$, parameter vector $\boldsymbol{\beta} = (\beta_1\,,\beta_2)'$ and design matrix,

$$\boldsymbol{X} = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}.$$

Note that

$$\boldsymbol{X}'\boldsymbol{X} = \begin{pmatrix} \sum_{i=1}^n a_i^2 & \sum_{i=1}^n a_ib_i \\ \sum_{i=1}^n a_ib_i & \sum_{i=1}^n b_i^2 \end{pmatrix},$$

so that

$$(\boldsymbol{X}'\boldsymbol{X})^{-1} = \frac{1}{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_ib_i\right)^2} \begin{pmatrix} \sum_{i=1}^n b_i^2 & -\sum_{i=1}^n a_ib_i \\ -\sum_{i=1}^n a_ib_i & \sum_{i=1}^n a_i^2 \end{pmatrix},$$

provided that $\boldsymbol{X}$ has full rank.[1] Since

$$\hat{\boldsymbol{\beta}} \sim \mathrm{N}_2\left(\boldsymbol{\beta}\,, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}\right),$$

we can read off the following:

$$\mathrm{Cov}\left(\hat\beta_1\,,\hat\beta_2\right) = -\frac{\sigma^2\sum_{i=1}^n a_ib_i}{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_ib_i\right)^2}.$$

This is zero if and only if $\sum_{i=1}^n a_ib_i = 0$. Because $\hat\beta_1$ and $\hat\beta_2$ are jointly normal, they are independent if and only if they are uncorrelated; that is, if and only if $\sum_{i=1}^n a_ib_i = 0$.

2. See page 65 of Lecture 9 in my lecture notes.

5. Since $\mathrm{Var}(\hat{\boldsymbol{Y}}) = \boldsymbol{X}\,\mathrm{Var}(\hat{\boldsymbol{\beta}})\boldsymbol{X}' = \sigma^2\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$, the variance of $\hat Y_i$ is the $(i\,,i)$th element of the preceding variance/covariance matrix; that is,

$$\mathrm{Var}(\hat Y_i) = \sigma^2\sum_{j=1}^p\sum_{k=1}^p X_{i,j}\left[(\boldsymbol{X}\boldsymbol{X})^{-1}\right]_{j,k} X_{i,k}.$$

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[1] In this case, this means that $(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) \neq (\sum_{i=1}^n a_ib_i)^2$. According to the Cauchy–Schwarz inequality, $(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) \geq (\sum_{i=1}^n a_ib_i)^2$. Therefore, $\boldsymbol{X}$ has full rank if and only if the Cauchy–Schwarz inequality is a strict inequality. This turns out to mean that $z_i := a_ib_i$ is not a linear function of i.

Because

$$\sum_{i=1}^{n} X_{i,j} X_{i,k} = [\boldsymbol{X}'\boldsymbol{X}]_{j,k} \, ,$$

it follows that

$$\sum_{i=1}^{n} \mathrm{Var}(\hat{Y}_i) = \sigma^2 \sum_{j=1}^{p} \sum_{k=1}^{p} \left[(\boldsymbol{X}\boldsymbol{X})^{-1}\right]_{j,k} [\boldsymbol{X}'\boldsymbol{X}]_{j,k} = \sigma^2 \mathrm{tr}\left((\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{X})\right) .$$

Because $(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{X}) = \boldsymbol{I}_{p\times p}$, its trace is p; this does the job.