# Math 6010
# Solutions to homework 3

1, p. 31. Notice that

$$\begin{aligned} Y_1 &= \rho Y_0 + \varepsilon_1, \\ Y_2 &= \rho^2 Y_0 + \rho\varepsilon_1 + \varepsilon_2, \\ Y_3 &= \rho^3 Y_0 + \rho^2\varepsilon_1 + \rho\varepsilon_2 + \varepsilon_3. \end{aligned}$$

That is, if we set

$$\boldsymbol{\varepsilon} := \begin{pmatrix} Y_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A} := \begin{pmatrix} \rho & 1 & 0 & 0 \\ \rho^2 & \rho & 1 & 0 \\ \rho^3 & \rho^2 & \rho & 1 \end{pmatrix},$$

than $\boldsymbol{Y} = (Y_1\,, Y_2\,, Y_3)'$ satisfies

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\varepsilon}.$$

(a) $\mathrm{Var}(\boldsymbol{Y}) = \boldsymbol{A}\mathrm{Var}(\boldsymbol{\varepsilon})\boldsymbol{A}'$, and $\mathrm{Var}(\boldsymbol{\varepsilon}) = \mathrm{diag}(\sigma_0^2\,, \sigma^2\,, \sigma^2\,, \sigma^2)$.

(b) Since $\boldsymbol{\varepsilon} \sim \mathrm{N}_4(\mathbf{0}\,, \mathrm{diag}[\sigma_0^2\,, \sigma^2\,, \sigma^2\,, \sigma^2])$, $\boldsymbol{Y}$ has a 4-dimensional multivariate normal distribution with mean vector zero and variance matrix as in part (a).

2, p. 31. Implicitly, we are assuming that $\boldsymbol{A}$ and $\boldsymbol{B}$ are not random. In that case, $[\boldsymbol{X}'\,, \boldsymbol{U}' + \boldsymbol{V}']'$ has a $2n$-dimensional multivariate normal distribution. Indeed,

$$\boldsymbol{W} := \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{U}' + \boldsymbol{V}' \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} + \boldsymbol{C} \end{pmatrix} \boldsymbol{Y}.$$

Therefore, it suffices to prove that $\boldsymbol{X}$ and $\boldsymbol{U} + \boldsymbol{V}$ are uncorrelated. But that follows because

$$\mathrm{Cov}[\boldsymbol{X}\,, \boldsymbol{U} + \boldsymbol{V}] = \mathrm{Cov}[\boldsymbol{X}\,, \boldsymbol{U}] + \mathrm{Cov}[\boldsymbol{X}\,, \boldsymbol{V}].$$

3, p. 31. Let $\boldsymbol{Y} = (Y_1\,, \ldots\,, Y_n)'$ and note that

$$\overline{Y} = \tfrac{1}{n}\mathbf{1}'\boldsymbol{Y},$$

where $\mathbf{1}$ is an $n$-vector of all ones. Also note that

$$\sum_{i=1}^{n-1}(Y_i - Y_{i+1})^2 = \|\boldsymbol{D}\|^2,$$

where $\boldsymbol{D}$ is the $(n-1)$-dimensional vector

$$\boldsymbol{D} := \begin{pmatrix} Y_1 - Y_2 \\ Y_2 - Y_3 \\ \vdots \\ Y_{n-1} - Y_n \end{pmatrix} = \boldsymbol{AY},$$

where

$$\boldsymbol{A} := \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

In particular, all linear combinations of $\overline{Y}$ and the coordindates of $\boldsymbol{D}$ are normally distributed [they are linear combinations of $Y_1, \ldots, Y_n$]. Therefore, it suffices to prove that $\boldsymbol{D}$ and $\overline{Y}$ are uncorrelated. Because $\mathrm{E}[\boldsymbol{D}] = \mathbf{0}$,

$$\mathrm{Cov}\left(\overline{Y}\,, \boldsymbol{D}\right) = \mathrm{Cov}\left(\tfrac{1}{n}\mathbf{1}'\boldsymbol{Y}\,, \boldsymbol{AY}\right) = \frac{1}{n^2}\mathbf{1}'\mathrm{Var}(\boldsymbol{Y})\boldsymbol{A}' = \frac{\sigma^2}{n^2}\mathbf{1}'\boldsymbol{A}' = \mathbf{0},$$

since the row-sums of $\boldsymbol{A}$ are all zero.

4, p. 31. All linear combinations of the coordinates are normal. Specifically, for all nonrandom vectors $\boldsymbol{t}$,

$$\boldsymbol{t}'[a\boldsymbol{X} + b\boldsymbol{Y}] = a\boldsymbol{t}'\boldsymbol{X} + b\boldsymbol{t}'\boldsymbol{Y}$$

is a sum of two independent normal random variables: $a\boldsymbol{t}'\boldsymbol{X}$ and $b\boldsymbol{t}'\boldsymbol{Y}$.