Math 6010
Solutions to homework 2

3, p. 15. Recall that $\sum_{i=1}^n (x_i - \bar{x})^2 = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ for every $\boldsymbol{x}$, where

$$\boldsymbol{A} := \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & 1-\frac{1}{n} \end{pmatrix}.$$

Therefore, $\mathrm{E}(Q_1) = \mathrm{tr}(\boldsymbol{A\Sigma}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$. Because $\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu} = \sum_{i=1}^n(\mu_i - \bar{\mu})^2$ and the μ_i's are all the same it follows that $\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu} = 0$ and therefore $\mathrm{E}(Q_1) = \mathrm{tr}(\boldsymbol{A\Sigma}) = \sum_i \sum_j A_{i,j}\Sigma_{i,j}$. Because $\Sigma_{i,j} = 0$ when $j > i+1$, and since $\boldsymbol{\Sigma}$ and $\boldsymbol{A}$ are both symmetric,

$$\begin{aligned}\mathrm{E}(Q_1) &= \sum_i \sum_j A_{i,j}\Sigma_{i,j} = \sum_{i=1}^n A_{i,i}\Sigma_{i,i} + 2\sum_{i=1}^n A_{i,i+1}\Sigma_{i,i+1} \\ &= \left(1 - \frac{1}{n}\right)\sum_{i=1}^n \Sigma_{i,i} - \frac{2}{n}\sum_{i=1}^n \Sigma_{i,i+1}.\end{aligned}$$

This does the job for Q_1. As regards Q_2, we appeal to Example 1.8 (p. 10) to see that

$$\begin{aligned}Q_2 &= 2\sum_{i=1}^n X_i^2 - X_1^2 - X_n^2 - 2\sum_{i=1}^{n-1} X_iX_{i+1} + (X_1 - X_n)^2 \\ &= 2\sum_{i=1}^n X_i^2 - 2\sum_{i=1}^{n-1} X_iX_{i+1} - 2X_1X_n.\end{aligned}$$

Because $\mathrm{E}(X_1X_n) = \mathrm{E}(X_1)\mathrm{E}(X_n) = \mu^2$ [I am assuming that $n > 3$; otherwise the result is silly], it follows that

$$\mathrm{E}(Q_2) = 2\sum_{i=1}^n \mathrm{E}(X_i^2) - 2\sum_{i=1}^{n-1} \mathrm{E}(X_iX_{i+1}) - 2\mu^2.$$

Now, $\mathrm{E}(X_i^2) = \mathrm{Var}(X_i) + (\mathrm{E}X_i)^2 = \Sigma_{i,i} + \mu^2$, and $\mathrm{E}(X_iX_{i+1}) = \Sigma_{i,i+1} + \mu^2$ similarly. Therefore,

$$\mathrm{E}(Q_2) = 2\sum_{i=1}^n \Sigma_{i,i} + 2n\mu^2 - 2\sum_{i=1}^{n-1} \Sigma_{i,i+1} - 2(n-1)\mu^2 - 2\mu^2 = 2\sum_{i=1}^n \Sigma_{i,i} - 2\sum_{i=1}^{n-1} \Sigma_{i,i+1}.$$

Therefore,

$$\begin{aligned}
\mathrm{E}(3Q_1 - Q_2) &= 3\left[\left(1-\frac{1}{n}\right)\sum_{i=1}^n \Sigma_{i,i} - \frac{2}{n}\sum_{i=1}^n \Sigma_{i,i+1}\right] - \left[2\sum_{i=1}^n \Sigma_{i,i} - 2\sum_{i=1}^{n-1}\Sigma_{i,i+1}\right]\\
&= \left(1-\frac{3}{n}\right)\sum_{i=1}^n \Sigma_{i,i} + \left(2-\frac{6}{n}\right)\sum_{i=1}^{n-1}\Sigma_{i,i+1}\\
&= \frac{n-3}{n}\sum_{i=1}^n \Sigma_{i,i} + 2\left(\frac{n-3}{n}\right)\sum_{i=1}^{n=1}\Sigma_{i,i+1}.
\end{aligned}$$

Divide both sides by $n(n-3)$ in order to see that

$$\mathrm{E}\left(\frac{3Q_1 - Q_2}{n(n-3)}\right) = \frac{1}{n^2}\sum_{i=1}^n \Sigma_{i,i} + \frac{2}{n^2}\sum_{i=1}^{n-1}\Sigma_{i,i+1}.$$

Finally, recall that

$$\mathrm{Var}(X_1 + \cdots + X_n) = \sum_{i=1}^n\sum_{j=1}^n \mathrm{Cov}(X_i\,, X_j) = \sum_{i=1}^n \Sigma_{i,i} + 2\sum_{i=1}^{n-1}\Sigma_{i,i+1}.$$

[The sum is the sum of the diagonal entries plus twice the sum of the entires that lie above the diagonal, by symmetry.] Therefore,

$$\mathrm{Var}(\bar{X}) = \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) = \frac{1}{n^2}\sum_{i=1}^n \Sigma_{i,i} + \frac{2}{n^2}\sum_{i=1}^{n-1}\Sigma_{i,i+1},$$

which as we have seen is equal to the expectation of $(3Q_1 - Q_2)/n(n-3)$.

4, p. 16. The matrix of the quadratic form $(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2$ is

$$\boldsymbol{A} := \begin{pmatrix} 2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}.$$

According to Theorem 1.6 (p. 10), the variance of $(X_1 - X_2)^2 + (X_2 - X_3)^2 + (X_1 - X_3)^2 = \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ is

$$(\mu_4 - 3\mu_2^2)\boldsymbol{a}'\boldsymbol{a} + 2\mu_2^2\,\mathrm{tr}(\boldsymbol{A}^2),$$

because the means [i.e., the θ's in that theorem] are all zero. Here, $\boldsymbol{a} := (2\,, 2\,, 2)'$ is the diagonals-vector of $\boldsymbol{A}$,

$$\mu_2 = \mathrm{E}(X_i^2) = \int_{-1}^1 \frac{1}{2}x^2\mathrm{d}x = \frac{1}{3},$$

and

$$\mu_4 = \mathrm{E}(X_i^4) = \int_{-1}^{1} \frac{1}{2} x^4 \mathrm{d}x = \frac{1}{5}.$$

Therefore, $(\mu_4 - 3\mu_2^2)\boldsymbol{a}'\boldsymbol{a} = -\frac{24}{15} = -\frac{8}{5}$. Since $\boldsymbol{A}$ is symmetric,

$$\mathrm{tr}(\boldsymbol{A}^2) = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{i,j}A_{i,j} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{i,j}^2 = 18.$$

Therefore, the answer to this question is $-\frac{8}{5} + 4 = \frac{12}{5}$.

5, p. 16. Observe that $\mathbf{X}'\mathbf{AX}$ and $\mathbf{X}'\mathbf{BX}$ are 1-dimensional random variables. Consequently,

$$\begin{aligned}\mathbf{X}'\mathbf{AX} \cdot \mathbf{X}'\mathbf{BX} &= \sum_{i=1}^{n}\sum_{j=1}^{n} X_i A_{i,j} X_j \cdot \sum_{k=1}^{n}\sum_{l=1}^{n} X_k B_{k,l} X_l \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} A_{i,j}B_{k,l}X_iX_jX_kX_l.\end{aligned}$$

Take expectations to see that

$$\begin{aligned}\mathrm{E}\left[\mathbf{X}'\mathbf{AX}\,, \mathbf{X}'\mathbf{BX}\right] &= \mathrm{E}\left(\mathbf{X}'\mathbf{AX} \cdot \mathbf{X}'\mathbf{BX}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} A_{i,j}B_{k,l}\mathrm{E}\left(X_iX_jX_kX_l\right).\end{aligned}$$

Because the X_i's are independent and have mean and third moment zero, all of these terms are zero except in the following cases: (i) When $i = j = k = l$; (ii) $i = j \neq k = l$; (iii) $i = k \neq j = l$; (iv) $i = l \neq j = k$. In case (i), $\mathrm{E}(X_iX_jX_kX_l) = \mathrm{E}(X_1^4) = 3\sigma^4$. In the remaining cases (i)–(iv), $\mathrm{E}(X_iX_jX_kX_l) = |\mathrm{E}(X_1^2)|^2 = \sigma^4$. Therefore,

$$\begin{aligned}\mathrm{E}\left[\mathbf{X}'\mathbf{AX}\,, \mathbf{X}'\mathbf{BX}\right] = 3\sigma^4 \sum_{i=1}^{n} A_{i,i}B_{i,i} &+ \sigma^4 \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n} A_{i,i}B_{k,k} \\ &+ \sigma^4 \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} A_{i,j}B_{i,j} + \sigma^4 \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} A_{i,j}B_{j,i},\end{aligned}$$

considering each case separately in order from (i)–(iv). Because $\mathbf{A}$ and $\mathbf{B}$ are symmetric, the last two quantities are equal. This reduces the computation to the following:

$$\begin{aligned}\mathrm{E}\left[\mathbf{X}'\mathbf{AX}\,, \mathbf{X}'\mathbf{BX}\right] = 3\sigma^4 \sum_{i=1}^{n} A_{i,i}B_{i,i} &+ \sigma^4 \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n} A_{i,i}B_{k,k} \\ &+ 2\sigma^4 \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} A_{i,j}B_{i,j}.\end{aligned}$$

Now,

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} A_{i,j}B_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}B_{i,j} - \sum_{i=1}^{n} A_{i,i}B_{i,i}$$

$$= \mathrm{tr}(\mathbf{AB}) - \sum_{i=1}^{n} A_{i,i}B_{i,i}.$$

Consequently,

$$\begin{aligned}\mathrm{E}\left[\mathbf{X}'\mathbf{A}\mathbf{X}\,,\mathbf{X}'\mathbf{B}\mathbf{X}\right] &= \sigma^4\sum_{i=1}^{n} A_{i,i}B_{i,i} + \sigma^4\sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n} A_{i,i}B_{k,k} \\ &\quad + 2\sigma^4\mathrm{tr}(\mathbf{AB}) \\ &= \sigma^4\sum_{i=1}^{n}\sum_{k=1}^{n} A_{i,i}B_{k,k} + 2\sigma^4\mathrm{tr}(\mathbf{AB}) \\ &= \sigma^4\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) + 2\sigma^4\mathrm{tr}(\mathbf{AB}) \\ &= \mathrm{E}\left(\mathbf{X}'\mathbf{A}\mathbf{X}\right)\mathrm{E}\left(\mathbf{X}'\mathbf{B}\mathbf{X}\right) + 2\sigma^4\mathrm{tr}(\mathbf{AB}),\end{aligned}$$

thanks to Theorem 1.5. This shows readily that

$$\mathrm{Cov}\left[\mathbf{X}'\mathbf{A}\mathbf{X}\,,\mathbf{X}'\mathbf{B}\mathbf{X}\right] = 2\sigma^4\mathrm{tr}(\mathbf{AB}).$$