Math 6010, Fall 2004: Homework

Homework 2

(1) Consider the set

$$S = \left\{ \boldsymbol{x} \in \mathbf{R}^n : \frac{1}{n}\sum_{i=1}^{n} x_i = 0 \right\}.$$

(a) Prove that S is a subspace of $\mathbf{R}^n$.

Solution. If $\boldsymbol{x}$ is in S and α is a real number, then the coordinates of $\alpha\boldsymbol{x}$ average to $\alpha\bar{\boldsymbol{x}} = 0$. Therefore, $\alpha\boldsymbol{x} \in S$. Furthermore, if $\boldsymbol{x}, \boldsymbol{y} \in S$, then $\overline{\boldsymbol{x}+\boldsymbol{y}} = \bar{\boldsymbol{x}} + \bar{\boldsymbol{y}} = 0$. Therefore, $\boldsymbol{x}, \boldsymbol{y} \in S$. Because $\mathbf{0} \in S$, this proves that S is a subspace of $\mathbf{R}^n$.

(b) Compute the projection matrices $\mathbf{P}_S$ and $\mathbf{I}_n - \mathbf{P}_S$. Use the latter expression to find an expression for the orthogonal complement to S; i.e.,

$$S^{\perp} = \{\boldsymbol{y} \in \mathbf{R}^n : \boldsymbol{y}'\boldsymbol{x} = 0 \text{ for all } \boldsymbol{x} \in S\}.$$

Solution. The typical $\boldsymbol{x}$ in S has the form

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ -x_1 - x_2 - \cdots - x_{n-1} \end{pmatrix}.$$

Define n-dimensional vectors,

$$\boldsymbol{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \ldots, \boldsymbol{V}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix}.$$

It is easy then to see that any $\boldsymbol{x} \in S$ has the form $x_1\boldsymbol{V}_1 + \cdots + x_{n-1}\boldsymbol{V}_{n-1}$.

Now write the basis-matrix $\boldsymbol{V}$:

$$\boldsymbol{V} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{n-1} \\ -\mathbf{1}'_{n-1} \end{pmatrix}.$$

$\boldsymbol{V}$ has $n-1$ columns and n rows; $\mathbf{I}_k$ is the $(k \times k)$-identity matrix, and $\mathbf{1}_k$ denotes a k-vector of all ones. Of course,

$$\boldsymbol{V}' = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{n-1} & -\mathbf{1}_{n-1} \end{pmatrix}.$$

So the $(n-1) \times (n-1)$-dimensional matrix $\boldsymbol{V}'\boldsymbol{V}$ is

$$\boldsymbol{V}'\boldsymbol{V} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

A little experimentation shows that

$$(\boldsymbol{V}\boldsymbol{V}')^{-1} = \frac{1}{n} \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & -1 & \cdots & -1 & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & n-1 \end{pmatrix}.$$

That is, the diagoal entries of the matrix $(\boldsymbol{V}'\boldsymbol{V})^{-1}$ are all $(\frac{n-1}{n})$, and the off-diagonals are all $-(\frac{1}{n})$. Now,

$$\boldsymbol{V}(\boldsymbol{V}'\boldsymbol{V})^{-1} = \frac{1}{n} \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & -1 & \cdots & -1 & -1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & n-1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}.$$

Therefore, $\mathbf{P}_S = \boldsymbol{V}(\boldsymbol{V}'\boldsymbol{V})^{-1}\boldsymbol{V}'$ is given by

$$\mathbf{P}_S = \frac{1}{n}\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & n-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & n-1 & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n-1 & -1 \\ -1-1 & -1 & -1 & \cdots & -1 & -1 & -1 & n-1 \end{pmatrix}.$$

That is, all the diagonal entries are $(\frac{n-1}{n})$, and the off-diagonal ones are $-(\frac{1}{n})$. If you think about it, you could possibly have guessed this matrix. From here, we obtain

$$\mathbf{I}_n - \mathbf{P}_S = \frac{1}{n}\begin{pmatrix} -1 & n-1 & n-1 & \cdots & n-1 & n-1 \\ n-1 & -1 & n-1 & \cdots & n-1 & n-1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & -1 \end{pmatrix}.$$

That is, all the diagonal entries are $-(\frac{1}{n})$, and the off-diagonal ones are $(\frac{n-1}{n})$. Now check that for all $\boldsymbol{x} \in \mathbf{R}^n$,

$$\mathbf{P}_S\boldsymbol{x} = \begin{pmatrix} x_1 - \bar{\boldsymbol{x}} \\ x_2 - \bar{\boldsymbol{x}} \\ \vdots \\ x_n - \bar{\boldsymbol{x}} \end{pmatrix} \quad \Rightarrow \quad (\mathbf{I}_n - \mathbf{P}_S)\boldsymbol{x} = \boldsymbol{x} - \mathbf{P}_S\boldsymbol{x} = \begin{pmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{x}} \\ \vdots \\ \bar{\boldsymbol{x}} \end{pmatrix}.$$

Because $S^\perp$ is the collection of all vectors of the form $\mathbf{P}_S\boldsymbol{x}$, this means that $\boldsymbol{x} \in S^\perp$ if and only if $x_1 = \cdots = x_n$.

(c) For all $\boldsymbol{y} \in \mathbf{R}^n$ compute, explicitly, the distance between $\boldsymbol{y}$ and the subspace S.

Solution. The answer is $\|\boldsymbol{y} - \mathbf{P}_S\boldsymbol{y}\| = \|(\mathbf{I}_n - \mathbf{P}_S)\boldsymbol{y}\|$. But we just saw that $(\mathbf{I}_n - \mathbf{P}_S)\boldsymbol{y}$ is just $\bar{\boldsymbol{y}}$ times an n-vector of all ones. Therefore, $\|(\mathbf{I}_n - \mathbf{P}_S)\boldsymbol{y}\| = \sqrt{n}|\bar{\boldsymbol{y}}| = |y_1 + \cdots + y_n|/\sqrt{n}$.

(2) Prove that $Q(x_1, x_2) = x_1 x_2$ is a quadratic form.

Solution. Check that $Q(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x}$, where

$$\boldsymbol{Q} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

(3) Problem 4, page 12.

Solution. (a) $\mathrm{Var}\bar{\boldsymbol{X}} = \mathrm{Var}(\sum_{i=1}^n X_i)/n^2$. But $\mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{Var}(X_i) + \sum_{i\neq j} \mathrm{Cov}(X_i, X_j) = n\sigma^2 + n(n-1)\rho\sigma^2$. Therefore,

$$\mathrm{Var}\bar{\boldsymbol{X}} = \frac{\sigma^2}{n}\left[1 + (n-1)\rho\right].$$

This quantity must be non-negative. Therefore, $1+(n-1)\rho \geq 0$. From this it follows that $\rho \geq -1/(n-1)$. That $\rho \leq 1$ is from Math. 5010; cf. Chebyshev inequality.

(b) We want $EQ^2 = \sigma^2$. But

$$EQ = a\sum_{i=1}^n EX_i^2 + bE\left[\left(\sum_{i=1}^n X_i\right)^2\right].$$

First off, $EX_i^2 = \mathrm{Var}X_i + (EX_i)^2 = \sigma^2 + \mu^2$. Likewise, $E[(\sum_{i=1}^n X_i)^2] = \mathrm{Var}(\sum_{i=1}^n X_i) + (E\sum_{i=1}^n X_i)^2 = n\sigma^2 + n(n-1)\rho\sigma^2 + n^2\mu^2$; cf. part (**a**). Therefore,

$$\begin{aligned} EQ &= an\sigma^2 + an\mu^2 + bn\sigma^2 + bn(n-1)\rho\sigma^2 + bn^2\mu^2 \\ &= n\sigma^2(a + b + b(n-1)\rho) + n\mu^2(a + bn). \end{aligned}$$

Because $\mu \neq 0$ it follows that $a + bn = 0$. This zeros out the coefficient of $n\mu^2$. The coefficient of $n\sigma^2$ must therefore be $\frac{1}{n}$. That is, $a + b\{1 + (n-1)\rho\} = \frac{1}{n}$. Plug in $b = -a/n$ to find that $a - (a/n)\{1 + (n-1)\rho\} = \frac{1}{n}$. This forces

$$a = \frac{1/n}{1 - (1/n)\{1 + (n-1)\rho\}} = \frac{1}{(n-1)(1-\rho)}.$$

Thus, also,

$$b = -\frac{a}{n} = -\frac{1}{n(n-1)(1-\rho)}.$$

Collect terms to obtain:

$$Q = \frac{\sum_{i=1}^n X_i^2 - \frac{1}{n}\left(\sum_{i=1}^n X_i\right)^2}{(n-1)(1-\rho)} = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{(n-1)(1-\rho)}.$$