Math 5090–001, Fall 2009
Solutions to the Midterm Exam

1. *Suppose $X_1, \ldots, X_n$ are i.i.d. according to a density [and/or mass function] $f(x\,;\theta)$, where θ is unknown. State, very carefully, the Neymann–Pearson lemma for the simple hypothesis $H_0 : \theta = \theta_0$ versus the simple alternative $H_a : \theta = \theta_a$.*

Solution: See the text.

2. *Suppose $X_1, \ldots, X_n$ are i.i.d. EXP(θ)'s, where $\theta > 0$ is unknown. That is, they have common density*

$$f(x\,;\theta) = \begin{cases} \dfrac{1}{\theta}\mathrm{e}^{-x/\theta} & \textit{if } x > 0, \\ 0 & \textit{if } x \le 0. \end{cases}$$

Find the form of the UMP test for $H_0 : \theta = 1$ versus $H_a : \theta > 1$. Justify your assertion carefully.

Solution: The likelihood ratio for H_0 versus $H_a : \theta = \theta_a$ $[\theta_a > 1]$ is

$$L = \theta_a^n \exp\left\{ -\left(1 - \frac{1}{\theta_a}\right) \sum_{j=1}^n X_j \right\} = \theta_a^n \exp\left\{ -n\left(1 - \theta_a^{-1}\right) \bar{X} \right\}.$$

This is a decreasing function of $t(\boldsymbol{X}) := \bar{X}$ because $1 - \theta_a^{-1} > 0$. Therefore, the Neymann–Pearson lemma provides us with a UMP test against the one-sided alternative $H_a : \theta > 1$. That is, reject H_0 when $\bar{X} \ge c$. [The constant c comes from a χ^2-table. In fact, $2n\bar{X}$ is $\chi^2(2n)$ under H_0.]

3. (*Consider the following density on the interval* $(0\,,1)$*:*

$$f(x\,,\theta) := \begin{cases} \theta x^{\theta-1} & \textit{if } 0 < x < 1, \\ 0 & \textit{otherwise,} \end{cases}$$

where $\theta > 0$ is unknown.

(a) *Show that* $-2\theta \ln X_1 \sim \chi^2(2)$. *You may use, without derivation, the fact that if* $Y \sim \chi^2(2n)$, *then its MGF is*

$$\mathrm{E}\left[\mathrm{e}^{tY}\right] = \begin{cases} \dfrac{1}{1-2t} & \text{if } t < \frac{1}{2}, \\ \infty & \text{otherwise.} \end{cases}$$

Solution: We compute the MGF of $-2\theta \ln X_1$ as follows:

$$\begin{aligned} M(t) &= \mathrm{E}\left[\mathrm{e}^{-t2\theta \ln X_1}\right] = \mathrm{e}\left[X_1^{-2\theta t}\right] \\ &= \theta \int_0^1 x^{-2\theta t} x^{\theta - 1}\,\mathrm{d}x \\ &= \frac{1}{1-2t} \qquad \text{if } t < \frac{1}{2}, \end{aligned}$$

and ∞ if $t \geq \frac{1}{2}$. This and the uniqueness theorem for MGF's together imply the result.

(b) *Use the result of part (a) to find a* $(1-\alpha)100\%$ *confidence interval for* θ. *You may use (a) even if you did not derive it.*

Solution: By **(a)**, $-2\theta \sum_{j=1}^n \ln X_j \sim \chi^2(2n)$. Therefore,

$$\mathrm{P}\left\{ \chi^2_{\alpha/2}(2n) \leq -2\theta \sum_{j=1}^n \ln X_j \leq \chi^2_{1-(\alpha/2)}(2n) \right\} = 1 - \alpha.$$

Solve algebraically, all the time recalling that $-\sum_{j=1}^n \ln X_j$ is a positive random variable, to see that

$$\mathrm{P}\left\{ \frac{\chi^2_{1-(\alpha/2)}(2n)}{-2\sum_{j=1}^n \ln X_j} \leq \theta \leq \frac{\chi^2_{\alpha/2}(2n)}{-2\sum_{j=1}^n \ln X_j} \right\} = 1 - \alpha.$$

Therefore, a $(1-\alpha)100\%$ CI for θ is

$$\left(\frac{\chi^2_{1-(\alpha/2)}(2n)}{-2\sum_{j=1}^n \ln X_j} \,,\, \frac{\chi^2_{\alpha/2}(2n)}{-2\sum_{j=1}^n \ln X_j} \right).$$