Math 5090–001, Fall 2009
Solutions to Assignment 9

Chapter 15, Problem 1. Routine [answers in back].

Chapter 15, Problem 8. (a) Theorem 15.3.1 tells us that

$$\mathrm{E}\hat\beta_0 = \beta_0, \qquad \mathrm{Var}\hat\beta_0 = \frac{\sigma^2 \sum_{i=1}^n X_i^2}{n\sum_{i=1}^n (X_i - \bar X)^2}.$$

Note that

$$\begin{aligned}\mathrm{Var}\hat\beta_0 &= \frac{\sigma^2}{n}\cdot\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n (X_i-\bar X)^2}\\ &= \frac{\sigma^2}{n}\left(1 - \frac{\sum_{i=1}^n (X_i-\bar X)^2 - \sum_{i=1}^n X_i^2}{\sum_{i=1}^n (X_i-\bar X)^2}\right).\end{aligned}$$

Because

$$\begin{aligned}\sum_{i=1}^n (X_i-\bar X)^2 &= \sum_{i=1}^n \left(X_i^2 - 2X_i\bar X + (\bar X)^2\right)\\ &= \sum_{i=1}^n X_i^2 - n(\bar X)^2,\end{aligned}$$

$$\mathrm{Var}\hat\beta_0 = \frac{\sigma^2}{n}\left(1 - \frac{n(\bar X)^2}{\sum_{i=1}^n (X_i-\bar X)^2}\right).$$

The result follows from this and the fact that $\mathrm{E}(\hat\beta_0^2) = \mathrm{Var}\hat\beta_0 + (\mathrm{E}\hat\beta_0)^2$.

(b) This follows directly from Theorem 15.3.1 and the fact that $\mathrm{E}(\hat\beta_1^2) = \mathrm{Var}\hat\beta_1 + (\mathrm{E}\hat\beta_1)^2$.

(c) First of all,

$$\sum_{i=1}^n (Y_i - \bar Y)^2 = \sum_{i=1}^n Y_i^2 - n(\bar Y)^2.$$

Therefore,

$$\mathrm{E}\sum_{i=1}^{n}(Y_i-\bar{Y})^2=\sum_{i=1}^{n}\mathrm{E}(Y_i^2)-n\mathrm{E}\left[(\bar{Y})^2\right].$$

Now,

$$\mathrm{E}(Y_i^2)=\mathrm{Var}(Y_i)+(\mathrm{E}Y_i)^2=\sigma^2+(\beta_0+\beta_1X_i)^2.$$

Therefore,

$$\sum_{i=1}^{n}\mathrm{E}(Y_i^2)=n\sigma^2+\sum_{i=1}^{n}(\beta_0+\beta_1X_i)^2.$$

Also,

$$\mathrm{E}\left[(\bar{Y})^2\right]=\mathrm{Var}(\bar{Y})+(\mathrm{E}\bar{Y})^2=\frac{\sigma^2}{n}+(\beta_0+\beta_1\bar{X})^2.$$

It follows that

$$\mathrm{E}\sum_{i=1}^{n}(Y_i-\bar{Y})^2=(n-1)\sigma^2+\sum_{i=1}^{n}Z_i^2-n(\bar{Z})^2,$$

where $Z_i:=\beta_0+\beta_1X_i$. Since

$$\sum_{i=1}^{n}Z_i^2-n(\bar{Z})^2=\sum_{i=1}^{n}(Z_i-\bar{Z})^2=\beta_1^2\sum_{i=1}^{n}(X_i-\bar{X})^2,$$

the result follows.

(d) First of all, note that

$$\begin{aligned}\sum_{i=1}^{n}(Y_i-\hat{\beta}_0-\hat{\beta}_1X_i)^2&=\sum_{i=1}^{n}\left(\left[Y_i-\bar{Y}\right]-\left[\hat{\beta}_0+\hat{\beta}_1X_i-\bar{Y}\right]\right)^2\\&=\sum_{i=1}^{n}(Y_i-\bar{Y})^2+\sum_{i=1}^{n}(\hat{\beta}_0+\hat{\beta}_1X_i-\bar{Y})^2-2\sum_{i=1}^{n}(Y_i-\bar{Y})(\hat{\beta}_0+\hat{\beta}_1X_i-\bar{Y}).\end{aligned}$$

Because $\bar{Y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}$ [p. 502], $\hat{\beta}_0 + \hat{\beta}_1 X_i - \bar{Y} = \hat{\beta}_1(X_i - \bar{X})$. Plug in the previous display to find that

$$\begin{aligned}
&\sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \\
&= \sum_{i=1}^n (Y_i - \bar{Y})^2 + \hat{\beta}_1^2 \sum_{i=1}^n (X_i - \bar{X})^2 - 2\hat{\beta}_1 \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X}) \\
&= \sum_{i=1}^n (Y_i - \bar{Y})^2 - \hat{\beta}_1 \sum_{i=1}^n (X_i - \bar{X})^2 \left[-\hat{\beta}_1 + 2 \frac{\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} \right]
\end{aligned}$$

Since (p. 502)

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2},$$

it follows that

$$\sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 = \sum_{i=1}^n (Y_i - \bar{Y})^2 - \hat{\beta}_1^2 \sum_{i=1}^n (X_i - \bar{X})^2.$$

Take expectations next:

$$\begin{aligned}
\mathrm{E} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 &= \overbrace{(n-1)\sigma^2 + \beta_1^2 \sum_{i=1}^n (X_i - \bar{X})^2}^{\text{by (c)}} - \mathrm{E}(\hat{\beta}_1^2) \sum_{i=1}^n (X_i - \bar{X})^2 \\
&= (n-1)\sigma^2 + \beta_1^2 \sum_{i=1}^n (X_i - \bar{X})^2 - \left\{ \beta_1^2 + \frac{\sigma^2}{\sum_{i=1}^n (X_i - \bar{X})^2} \right\} \sum_{i=1}^n (X_i - \bar{X})^2,
\end{aligned}$$

by (b). Therefore, it follows that

$$\mathrm{E} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 = (n-2)\sigma^2,$$

which is the claim.