Math 5090–001, Fall 2009

Solutions to Assignment 5

Chapter 12, Problem 10. The company sizes are very large [$n_1 = n_2 = 200$ each]. Therefore, we can use an approximate [large-sample] test for $H_0: \ p_1 = p_2$ versus $H_a: \ p_1 \neq p_2$ at level $\alpha = 0.05$.

Let $\hat{p}_1$ and $\hat{p}_2$ denote respectively the proportion of nondefectives in the samples from company 1 and 2. [In our sample, $\hat{p}_1 = 180/400$ and $\hat{p}_2 = 190/400$].

Since $n_1 = n_2 = 200$ is large, we apply large-sample asymptotics and reject when

$$Z := \frac{|\hat{p}_1 - \hat{p}_2|}{\sqrt{(n_1\hat{p}_1 + n_2\hat{p}_2)\left(1 - \frac{n_1\hat{p}_1+n_2\hat{p}_2}{n_1+n_2}\right)}} \approx 1.898 \quad \text{is} > z_{1-(\alpha/2)}{=}1.96.$$

So we do not reject H_0 at $\alpha = 0.05$.

Chapter 12, Problem 11. **(a)** We compute the likelihood ratio test according to the Neymann-Pearson lemma. Namely, we reject H_0 if the likelihood under H_0 is much less than that under H_a; that is, when

$$L = \frac{\theta_0 X_1^{\theta_0-1}}{\theta_1 X_1^{\theta_1-1}} = \frac{1}{2X_1} \qquad \text{is small.}$$

Since $X_1 > 0$, L is small if and only if X_1 is large. So we reject when $X_1 > c$. In order to find c, we set

$$0.05 = \alpha = \mathrm{P}\left\{X_1 > c \,\middle|\, \theta = 1\right\} = \int_c^1 \mathrm{d}x = 1 - c.$$

Therefore, $c = 0.95$, and we reject H_0 when $X_1 > 0.95$.

(b) The power again H_a is

$$\mathrm{P}\left\{X_1 > 0.95 \,\middle|\, \theta = 2\right\} = \int_{0.95}^1 2x\,\mathrm{d}x = 1 - 0.95^2 = 0.0975.$$

(c) As before, we reject when L is small, where

$$L = \frac{\theta_0^n X_1^{\theta_0 - 1} \cdots X_n^{\theta_0 - 1}}{\theta_1^n X_1^{\theta_1 - 1} \cdots X_n^{\theta_1 - 1}} = \frac{1}{2^n X_1 \cdots X_n} \quad \text{is small.}$$

Equivalently, reject H_0 when $X_1 \cdots X_n = \exp\{\sum_{j=1}^n \ln X_j\}$ is large. Yet equivalently, we reject H_0 when $\frac{1}{n}\sum_{j=1}^n \ln X_j$ is large. It is more convenient to work with positive numbers; therefore,

$$\text{we reject } H_0 \text{ when } -\sum_{j=1}^n \ln X_j < c.$$

Therefore, we need to find the distribution of $-\sum_{j=1}^n \ln X_j$ under H_0.

Note that the moment generating function of $-2\ln X_1$, under H_0, is easy to find:

$$\begin{aligned} M(t) &= \mathrm{E}\left(\mathrm{e}^{-2t \ln X_1}\right) = \mathrm{E}\left(\frac{1}{X_1^{2t}}\right) = \int_0^1 x^{-2t}\, \mathrm{d}x \\ &= \begin{cases} \frac{1}{1-2t} & \text{if } t < 1/2, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

That is, under H_0, $-2\ln X_1 \sim \chi^2(2)$; and hence $-2\sum_{j=1}^n \ln X_j \sim \chi^2(2n)$. So,

$$\alpha = \mathrm{P}\left\{ -2\sum_{j=1}^n \ln X_j < 2c \;\middle|\; \theta = 1 \right\} = \mathrm{P}\left\{\chi^2(2n) < 2c\right\};$$

and hence $2c = \chi^2_\alpha(2n)$. In other words,

$$\text{we reject } H_0 \text{ when } -\sum_{j=1}^n \ln X_j < \frac{1}{2}\chi^2_\alpha(2n).$$

Chapter 12, Problem 15. The N-P lemma tells us that we should reject H_0 when

$$L := \frac{f(\boldsymbol{X};\theta_0)}{f(\boldsymbol{X};\theta_1)} < c.$$

According to the factorization theorem [p. 339], the sufficiency of S implies that we can write

$$L = \frac{g(S\,;\theta_0)h(\boldsymbol{X})}{g(S\,;\theta_1)h(\boldsymbol{X})} = \frac{g(S\,;\theta_0)}{g(S\,;\theta_1)}.$$

Therefore, we reject when the latter—which depends through the data only via S—is small.

Chapter 12, Problem 16. The likelihood ratio, for H_0 against $H_a:\ \theta = \theta_a$—where $\theta_a > \theta_0$—is

$$L := \left(\frac{\theta_a}{\theta_0}\right)^n \exp\left\{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{j=1}^n X_j^3\right\} < c.$$

This is a monotone likelihood ratio [Definition 12.7.2, p. 413] with $t(\boldsymbol{X}) := \sum_{j=1}^n X_j^3$. Therefore [Theorem 12.7.1, p. 414], the UMP test for $H_0:\ \theta = \theta_0$ versus $H_a:\ \theta > \theta_0$ is:

$$\text{Reject } H_0 \text{ when } \sum_{j=1}^n X_j^3 > c.$$

And of course c is computed via:

$$\alpha = \mathrm{P}\left\{\sum_{j=1}^n X_j^3 > c \;\middle|\; \theta = \theta_0\right\}.$$

So, let us compute c via distribution theory: First of all, for all $a > 0$,

$$F_{2X_1^3}(a) = \mathrm{P}\left\{2X_1^3 \le a\right\} = \mathrm{P}\left\{X_1 \le (a/2)^{1/3}\right\} = F_{X_1}\left((a/2)^{1/3}\right).$$

And $F_{2X_1^3}(a) = 0$ if $a \leq 0$. Differentiate [d/da]:

$$f_{2X^3}(a) = f_{X_1}\left((a/2)^{1/3}\right) \times \frac{\mathrm{d}}{\mathrm{d}a}\left((a/2)^{1/3}\right) = \frac{1}{2\theta}\mathrm{e}^{-x/(2\theta)} \quad \text{for } a > 0.$$

In other words, $2X_1^3, \ldots, 2X_n^3$ are i.i.d. EXP(2θ)'s. Equivalently, the sequence $2X_1^3/\theta, , \ldots 2X_n^3/\theta$ is one of all i.i.d. EXP(2)'s; and these exponentials are the same as $\chi^2(2)$'s [check MGFs in the back of the front cover]. This means that $(2/\theta_0)\sum_{j=1}^n X_j^3 \sim \chi^2(2n)$ under H_0. So we can write

$$\alpha = \mathrm{P}\left\{\frac{2}{\theta_0}\sum_{j=1}^n X_j^3 > \frac{2c}{\theta_0} \;\middle|\; \theta = \theta_0\right\} = \mathrm{P}\left\{\chi^2(2n) > \frac{2c}{\theta_0}\right\}.$$

And this means that $(2/\theta_0) = \chi^2_{1-\alpha}(2n)$. In other words:

$$\text{Reject } H_0 \text{ when} \quad \sum_{j=1}^n X_j^3 > \frac{1}{2}\theta_0\,\chi^2_{1-\alpha}(2n).$$