Math 5090–001, Fall 2009

Solutions to Assignment 4

Chapter 12, Problem 9. **(a)** If the variances are unknown but equal, then we use the pooled-variance method [see part 3 of Theorem 12.3.5 on p. 403]: The test is to reject H_0 when

$$\frac{|\bar{Y}-\bar{X}|}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \geq t_{1-(\alpha/2)}(n_1+n_2-2).$$

Now, for our data,

$$S_p = \sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}} = \sqrt{\frac{(8\times 36)+(8\times 45)}{9+9-2}} = \sqrt{40.5} \approx 6.36396.$$

Therefore, for our data set,

$$\frac{|\bar{Y}-\bar{X}|}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{|10-16|}{6.3696\times\sqrt{\frac{1}{9}+\frac{1}{9}}} \approx 1.998.$$

Since $t_{1-(\alpha/2)}(n_1+n_2-2) = t_{0.95}(16) \approx 1.746$. Because $1.998 > 1.746$, we reject H_0 at level $\alpha = 0.1$.

(b) If we use Welch's approximation (11.5.13) on page 380, then we reject H_0 when

$$\frac{|\bar{Y}-\bar{X}|}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} > t_{0.95}(\nu),$$

where

$$\nu = \frac{\left(\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}\right)^2}{\left[\frac{(S_1^2/n_1)^2}{(n_1-1)}+\frac{(S_2^2/n_2)^2}{(n_2-1)}\right]} = \frac{\left(\frac{36}{9}+\frac{45}{9}\right)^2}{\left[\frac{(36/9)^2}{8}+\frac{(45/9)^2}{8}\right]} \approx 16 \Rightarrow t_{0.95}(\nu) \approx 1.746.$$

In our data set,

$$\frac{|\bar{Y}-\bar{X}|}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} = \frac{|10-16|}{\sqrt{\frac{36}{9}+\frac{45}{9}}} = 2.$$

This is greater than 1.746, so we reject H_0 in this way, as well. Note that in this case, we are *not* assuming a priori that $\sigma_1 = \sigma_2$.

(c) If we know that this is in fact paired data with $S_D = 9$ and $n = n_1 = n_2 = 9$, then we use (11.5.17) from page 381 to find that we reject H_0 when

$$\frac{|\bar{Y} - \bar{X}|}{S_D/\sqrt{n}} > t_{0.95}(n-1) \approx 1.86.$$

For our data set,

$$\frac{|\bar{Y} - \bar{X}|}{S_D/\sqrt{n}} = \frac{|10 - 16|}{\sqrt{81}/\sqrt{9}} = 2.$$

This is greater than 1.86. So we reject in the paired-sample case.

(d) According to Theorem 12.3.4 of page 402, we reject H_0 if

$$F = \frac{S_1^2}{S_2^2} > f_{1-\alpha}(n_1 - 1\,, n_2 - 1) = f_{0.95}(8\,,8) \approx 3.44.$$

But for our data set $F = 36/45 = 0.8$. Therefore, we do not reject H_0.

(e) We need to compute

$$\texttt{power at } 1.33 = \mathrm{P}\left\{ \frac{S_1^2}{S_2^2} > 3.44 \,\middle|\, \frac{\sigma_1^2}{\sigma_2^2} = 1.33 \right\}.$$

If $\sigma_1^2/\sigma_2^2 = 1.33$, then

$$F := \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2} \cdot \frac{1}{1.33}, \quad \text{and } F \sim F(8\,,8).$$

Therefore,

$$\texttt{power at } 1.33 = \mathrm{P}\left\{ F \times 1.33 > 3.44 \,\middle|\, \frac{\sigma_1^2}{\sigma_2^2} = 1.33 \right\} = \mathrm{P}\left\{ F(8\,,8) > \underbrace{\frac{3.44}{1.33}}_{\approx 2.59} \right\}.$$

Therefore, the power is 0.1 against $H_a : \sigma_1^2/\sigma_2^2 = 1.33$.

Chapter 12, Problem 31. First of all, note that this is *not* a Paréto distribution, but related closely to one. A second notable remark is that the parameter space has to be the collection of all $\theta > 0$.

Now, the GLR tells us to reject when λ is large, where

$$\begin{aligned}
\lambda := \frac{f(\boldsymbol{X};\theta_0)}{\max_{\theta\neq\theta_0} f(\boldsymbol{X};\theta)} &= \frac{\theta_0^n \prod_{j=1}^n X_j^{\theta_0-1}}{\hat\theta^n \prod_{j=1}^n X_j^{\hat\theta-1}} \\
&= \left(\frac{\theta_0}{\hat\theta}\right)^n \exp\left\{(\theta_0-\hat\theta)\sum_{j=1}^n \log X_j\right\} \\
&= \exp\left\{(\theta_0-\hat\theta)\sum_{j=1}^n \log X_j + n\log\left(\frac{\theta_0}{\hat\theta}\right)\right\},
\end{aligned}$$

where $\hat\theta$ is the MLE for θ [restricted to $\theta \neq \theta_0$].

Next, we find the MLE by maximizing the following likelihood function over $\theta \neq \theta_0$:

$$\mathcal{L}(\theta) := \theta^n \prod_{j=1}^n X_j^{\theta-1} = \exp\left\{n\log\theta + (\theta-1)\sum_{j=1}^n \log X_j\right\}.$$

Equivalently, we maximize $\log\mathcal{L}(\theta) = n\log\theta + (\theta-1)\sum_{j=1}^n \log X_j$ [this is the the log-likelihood]. Now,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\log\mathcal{L}(\theta) = \frac{n}{\theta} + \sum_{j=1}^n \log X_j, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log\mathcal{L}(\theta) = -\frac{n}{\theta^2} < 0.$$

Therefore, the MLE is

$$\hat\theta = -\frac{n}{\sum_{j=1}^n \log X_j}.$$

This is sensible because $0 < X_j < 1$, so that $\log X_j < 0$. This is also the MLE among all $\theta \neq \theta_0$, since $\mathrm{P}\{\hat\theta = \theta_0\} = 0$ [$\hat\theta$ has a pdf]. Plug to find that

$$\lambda = \exp\left\{-n\left(\frac{\theta_0}{\hat\theta} - 1\right) + n\log(\theta_0/\hat\theta)\right\}.$$

Because θ_0 is a fixed positive constant, we reject when

$$\lambda_1 := -n\left[\left(\frac{\theta_0}{\hat\theta} - 1\right) - \log(\theta_0/\hat\theta)\right]$$

is large. I.e., reject when

$$\lambda_2 := \left(\frac{\theta_0}{\hat\theta} - 1\right) - \log(\theta_0/\hat\theta)$$

is small.

Next we want to do asymptotics [this portion is not graded] for $n \to \infty$. Since $1/\hat\theta$ is the average of n i.i.d. random variables, it is approximately normal with mean

$$\mu := -\mathrm{E}\log X_1 = -\theta\int_0^1 \log(x)x^{\theta-1}\,dx = \theta\int_0^\infty ye^{-\theta y}\,dy \qquad [y := -\log x]$$
$$= \frac{1}{\theta}.$$

and variance

$$\begin{aligned}\frac{1}{n}\mathrm{Var}\,(\log X_1) &= \frac{1}{n}\left\{\mathrm{E}\left(|\log X_1|^2\right) - \frac{1}{\theta^2}\right\} = \frac{1}{n}\left\{\theta\int_0^1 (\log x)^2 x^{\theta-1}\,dx - \frac{1}{\theta^2}\right\}\\ &= \frac{1}{n}\left\{\theta\int_0^\infty y^2e^{-\theta y}\,dy - \frac{1}{\theta^2}\right\} = \frac{1}{n\theta^2}.\end{aligned}$$

Therefore, under H_0,

$$\sqrt{n}\,\theta_0\left(\frac{1}{\hat\theta} - \frac{1}{\theta_0}\right) \approx \mathrm{N}(0\,,1).$$

And this is another way of saying that

$$\frac{\theta_0}{\hat\theta} - 1 \approx \frac{1}{\sqrt{n}}\mathrm{N}(0\,,1). \tag{1}$$

Now we return to the test. First, by Taylor–McLaurin expansion,

$$\ln v \approx (v-1) - \frac{1}{2}(v-1)^2 \qquad \text{for } v \approx 1.$$

Or equivalently,

$$v - 1 - \log v \approx \frac{1}{2}(v-1)^2.$$

Since $\hat{\theta} \approx \theta_0$ [under H_0] with high probab., it follows from a 2-term Taylor expansion of the log that

$$\lambda_2 \approx \frac{1}{2}\left(\frac{\theta_0}{\hat{\theta}} - 1\right)^2$$

Therefore, (1) implies that

$$\lambda_2 \approx \frac{1}{2n}\chi^2(1).$$

That is, $2n\lambda_2$ is approximately a central $\chi^2(1)$. So approximately: Reject H_0 when $2n\lambda_2 \leq \chi^2_\alpha(1)$.

An Aside on the MLE. Let $g(x) := 1/x$ to find from Taylor expansion that

$$\hat{\theta} - \theta_0 = g(1/\hat{\theta}) - g(1/\theta_0) \approx \left(\frac{1}{\hat{\theta}} - \frac{1}{\theta_0}\right) g'(1/\theta_0) = -\theta_0^2 \left(\frac{1}{\hat{\theta}} - \frac{1}{\theta_0}\right).$$

Since the right-most bracketed term is asymptotically normal with mean zero and variance $(n\theta_0^2)^{-1}$, we find that

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right) \approx -\mathrm{N}(0\,,\theta_0^2) = \mathrm{N}(0\,,\theta_0^2).$$