Math 5090–001, Fall 2009
Solutions to Assignment 3

Chapter 12, Problem 1. **(a)** We want to find a and b such that

$$P\{\bar{X} \le a \,|\, \mu = 20\} = P\{\bar{X} \ge b \,|\, \mu = 20\} = 0.05.$$

If $\mu = 20$, then $\bar{X} \sim \text{N}(20\,, \sigma^2 = 1/16)$. Therefore,

$$0.05 = P\{\bar{X} \le a \,|\, \mu = 20\} = P\left\{\text{N}(0\,,1) \le \frac{a-20}{1/4}\right\}.$$

Therefore,

$$\frac{a-20}{1/4} \approx -1.645 \quad \Rightarrow \quad a \approx 19.58875.$$

Similarly,

$$0.05 = P\{\bar{X} \ge b \,|\, \mu = 20\} = P\left\{\text{N}(0\,,1) \ge \frac{b-20}{1/4}\right\}.$$

Therefore,

$$\frac{b-20}{1/4} \approx 1.645 \quad \Rightarrow \quad b \approx 20.41125.$$

(b) We are asked to find $\beta = P\{\text{not reject} \,|\, \mu = 21\}$ in both cases. For the critical region A:

$$\beta = P\{\bar{X} \ge 19.58875 \,|\, \mu = 21\} = P\left\{\text{N}(0\,,1) \ge \frac{19.58875 - 21}{1/4}\right\} \approx 1.$$

This is awful. And for the region B:

$$\beta = P\{\bar{X} \le 20.41125 \,|\, \mu = 21\} = P\left\{\text{N}(0\,,1) \le \frac{20.41125 - 21}{1/4}\right\} \approx 0.01.$$

Excellent!

(c) We are asked to find $\beta = P\{\text{not reject} \,|\, \mu = 19\}$ in both cases. For the critical region A:

$$\beta = P\{\bar{X} \geq 19.58875 \,|\, \mu = 19\} = P\left\{\mathrm{N}(0\,,1) \geq \frac{19.58875 - 19}{1/4}\right\} \approx 0.01.$$

This is excellent. And for the region B:

$$\beta = P\{\bar{X} \leq 20.41125 \,|\, \mu = 19\} = P\left\{\mathrm{N}(0\,,1) \leq \frac{20.41125 - 19}{1/4}\right\} \approx 1.$$

This is awful!

(d) We are asked to compute

$$\begin{aligned}
\alpha &= 1 - P\{19.58875 \leq \bar{X} \leq 20.41125 \,|\, \mu = 20\} \\
&= 1 - P\left\{\frac{19.58875 - 20}{1/4} \leq \mathrm{N}(0\,,1) \leq \frac{20.41125 - 20}{1/4}\right\} \\
&= 1 - P\{-1.645 \leq \mathrm{N}(0\,,1) \leq 1.645\} \\
&\approx 0.1.
\end{aligned}$$

(e) We are asked to find two probabilities: First,

$$\begin{aligned}
P\{19.58875 \leq \bar{X} \leq 20.41125 \,|\, \mu = 19\} &= P\left\{\frac{19.58875 - 19}{1/4} \leq \mathrm{N}(0\,,1) \leq \frac{20.41125 - 19}{1/4}\right\} \\
&= P\{2.355 \leq \mathrm{N}(0\,,1) \leq 5.645\} \\
&\approx 0.003;
\end{aligned}$$

and then

$$\begin{aligned}
P\{19.58875 \leq \bar{X} \leq 20.41125 \,|\, \mu = 21\} &= P\left\{\frac{19.58875 - 21}{1/4} \leq \mathrm{N}(0\,,1) \leq \frac{20.41125 - 21}{1/4}\right\} \\
&= P\{-5.645 \leq \mathrm{N}(0\,,1) \leq -2.355\} \\
&\approx 0.003.
\end{aligned}$$

That is,

$$P\left\{\text{not reject} \,\middle|\, |\mu - 20| = 1\right\} \approx 0.003.$$

Chapter 12, Problem 2. **(a)** We want

$$\begin{aligned}\alpha = P\{\text{reject} \,|\, \theta = 2\} &= P\{\text{both the same color} \,|\, \theta = 2\} \\ &= P\{\text{both white} \,|\, \theta = 2\} + P\{\text{both black} \,|\, \theta = 2\} \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.\end{aligned}$$

[This isn't a very good test, is it?]

(b) There are 4 situations: $\theta = 0, 1, 3, 4$.

$$\begin{aligned}P\{\text{not reject} \,|\, \theta = 0\} &= 0 \\ P\{\text{not reject} \,|\, \theta = 1\} &= P\{W_1 \cap B_2\} + P\{B_1 \cap W_2\} \\ &= \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right) = \frac{3}{8}. \\ P\{\text{not reject} \,|\, \theta = 3\} &= \frac{3}{8}. \\ P\{\text{not reject} \,|\, \theta = 4\} &= 0.\end{aligned}$$

(c) For α we want

$$\begin{aligned}\alpha = P\{\text{reject} \,|\, \theta = 2\} &= P\{\text{both white} \,|\, \theta = 2\} + P\{\text{both black} \,|\, \theta = 2\} \\ &= \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{2} \times \frac{1}{3}\right) = \frac{1}{3}.\end{aligned}$$

For β there are 4 situations: $\theta = 0, 1, 3, 4$.

$$\begin{aligned}P\{\text{not reject} \,|\, \theta = 0\} &= 0 \\ P\{\text{not reject} \,|\, \theta = 1\} &= \left(\frac{1}{4} \times 1\right) + \left(\frac{3}{4} \times \frac{1}{3}\right) = \frac{1}{2} \\ P\{\text{not reject} \,|\, \theta = 3\} &= \frac{1}{2} \\ P\{\text{not reject} \,|\, \theta = 4\} &= 0.\end{aligned}$$

Chapter 12, Problem 6. **(a)** We want to find c such that

$$\begin{aligned}\alpha &= P\{X_{1:n} \geq c \,|\, \eta = \eta_0\} \\ &= P\{X_1 \geq c\,, \ldots, X_n \geq c \,|\, \eta = \eta_0\} \\ &= P\{X_1 \geq c \,|\, \eta = \eta_0\} \times \cdots \times P\{X_n \geq c \,|\, \eta = \eta_0\} \\ &= \mathrm{e}^{-n(c-\eta_0)}\end{aligned}$$

Solve to get

$$c = \eta_0 + \frac{1}{n} \ln\left(\frac{1}{\alpha}\right).$$

(b) We are asked to compute the following for all $\eta_1 > \eta_0$:

$$\begin{aligned}P\{X_{1:n} \leq c \,|\, \eta = \eta_1\} &= 1 - P\{X_{1:n} \geq c \,|\eta = \eta_1\} \\ &= 1 - e^{-n(c-\eta_1)}.\end{aligned}$$

But c solves $e^{-n(c-\eta_0)} = \alpha$. Therefore,

$$\begin{aligned}P\{X_{1:n} \leq c \,|\, \eta = \eta_1\} &= 1 - e^{-n(c-\eta_0)} e^{-n(\eta_1-\eta_0)} \\ &= 1 - \alpha e^{-n(\eta_1-\eta_0)}.\end{aligned}$$

(c) We know from **(b)** that if $\eta = \eta_1$, then

$$\beta = \alpha e^{-n(\eta_1-\eta_0)} \quad \Rightarrow \quad n = \frac{\ln(\alpha/\beta)}{\eta_1 - \eta_0}.$$

But this is typically not an integer. Therefore, we choose conservatively; i.e.,

$$n \geq 1 + \left\lfloor \frac{\ln(\alpha/\beta)}{\eta_1 - \eta_0} \right\rfloor,$$

where $\lfloor \cdots \rfloor$ denotes the lowest-integer part function.