Math 5090–001, Fall 2009

Solutions to Assignment 2

Chapter 11, Problem 17. **(a)** The cdf of $X \sim \text{GEO}(p)$ is

$$G(s\,;p) = \sum_{j=1}^{s} p(1-p)^{j-1} = p\sum_{i=0}^{s-1}(1-p)^i = p\left[\frac{1-(1-p)^s}{1-(1-p)}\right] = 1-(1-p)^s,$$

for $s = 1, 2, \ldots$. Viewed as a function of p, $G(s\,;p)$ is increasing for every integer $s = 1, 2, \ldots$. In fact,

$$\frac{\partial}{\partial p} G(s\,;p) = s(1-p)^{s-1} \geq 0.$$

Therefore, the functions h_1 and h_2 of Theorem 11.4.3 are decreasing. A conservative one-sided $100(1-\alpha)\%$ lower CI for p is $(\theta_L\,, 1]$, where we find θ_L by solving $G(X;\theta_L) = \alpha$. That is, find θ_L such that

$$1-(1-\theta_L)^X = \alpha \quad \Leftrightarrow \quad \theta_L = 1-(1-\alpha)^{1/X}.$$

(b) The answer becomes $\theta_L = 1 - 0.9^{1/5} \approx 0.02085$. So the confidence interval is $\approx (0.029\,, 1]$.

(c) The sum of n independent GEO(p)'s is negative binomial; i.e.,

$$G(s\,;p) = \sum_{k=n}^{s} \binom{k-1}{n-1} p^n (1-p)^{k-n},$$

for $s \geq n$, an integer. If s is fixed, then

$$\frac{\partial}{\partial p} G(s\,;p) = \sum_{k=n}^{s} \binom{k-1}{n-1} \left\{ np^{n-1}(1-p)^{k-n} + (k-n)p^n(1-p)^{k-n-1} \right\} \geq 0.$$

So, once again, $G(s\,, p)$ increases with p, for every fixed integer $s \geq n$. And h_1 and h_2 are decreasing; so the CI is $(\theta_L\,, 1]$, where θ_L is the solution to $G(S\,;\theta_L) = \alpha$, where $S := X_1 + \cdots + X_n$ is the sum of the n i.i.d. geometrics.

Chapter 11, Problem 27. **(a)** [*I graded only this part.*] The length of the CI is

$$\left\{\Phi^{-1}(\alpha_1) - \Phi^{-1}(1-\alpha_2)\right\} \cdot \frac{\sigma}{\sqrt{n}}$$

Our goal is to minimize the preceding function of $\alpha_1, \alpha_2 \geq 0$, subject to $\alpha_1 + \alpha_2 = \alpha$. Equivalently, we must minimize

$$h(\alpha_1\,, \alpha - \alpha_1) := \Phi^{-1}(\alpha_1) - \Phi^{-1}(1-\alpha+\alpha_1) \quad \text{over all } \alpha_1 \in (0\,,1).$$

Now, elementary calculus tells us that

$$\begin{aligned}\frac{\partial}{\partial \alpha_1} h(\alpha_1\,, \alpha - \alpha_1) &= \frac{\partial}{\partial \alpha_1}\Phi^{-1}(\alpha_1) - \frac{\partial}{\partial \alpha_1}\Phi^{-1}(1-\alpha+\alpha_1) \\ &= \frac{1}{\phi\left(\Phi^{-1}(\alpha_1)\right)} - \frac{1}{\phi\left(\Phi^{-1}(1-\alpha+\alpha_1)\right)},\end{aligned}$$

where ϕ is the N$(0\,,1)$ density function. Set the preceding to zero to find that the optimal choice of α_1 solves

$$\phi\left(\Phi^{-1}(\alpha_1)\right) = \phi\left(\Phi^{-1}(1-\alpha+\alpha_1)\right).$$

Inspect the curve ϕ to see that if $\phi(a) = \phi(b)$ and $a \neq b$, then $a = -b$, necessarily! This means that

$$\Phi^{-1}(\alpha_1) = -\Phi^{-1}(1-\alpha+\alpha_1).$$

In other words [check the graph of Φ], $\alpha_1 = \alpha - \alpha_1$, and hence $\alpha_1 = \alpha/2$. In order to complete this computation, you need to show that the second derivative of $h(\alpha_1\,, \alpha - \alpha_1)$ [with respect to α_1] is positive at $\alpha_1 = \alpha/2$. I will let you do this yourself.

(b) Yes; replace σ by S and $\Phi^{-1}(\alpha_j)$ by $F^{-1}(\alpha_j)$, where F denotes the cdf of a t distribution with $n-1$ df. In this way, we find that the optimal α_1 solves

$$F'\left(F^{-1}(\alpha_1)\right) = F'\left(F^{-1}(1-\alpha+\alpha_1)\right).$$

Since the shape of F is similar to Φ, it has the same properties, and so $\alpha_1 = \alpha/2$ follows from the same considerations as before [you should check the details!].

(c) Because $E(S^2) = \sigma^2$, the expected length of the general form of the CI (11.3.6) is

$$2(n-1)\sigma^2 \left[\frac{1}{q_1} - \frac{1}{q_2}\right],$$

where $F(q_1) = \alpha_1$ and

$$F(q_2) = \alpha_2 = 1 - \alpha + \alpha_1 = \text{constant} + F(q_1), \qquad (\star)$$

and F is the cdf of a $\chi^2(n-1)$. Let $f := F'$, as usual. We can differentiate $(\star)$ $[d/dq_1]$ to find that

$$f(q_2)\frac{dq_2}{dq_1} = f(q_1) \quad \Rightarrow \quad \frac{dq_2}{dq_1} = \frac{f(q_1)}{f(q_2)}.$$

But the expected length is proportional to

$$h(q_1\,, q_2) = \frac{1}{q_1} - \frac{1}{q_2};$$

in fact, we saw that the constant of proportionality is $2(n-1)\sigma^2$. Want to set $\partial h/\partial q_1 = 0$;

$$\frac{\partial h}{\partial q_1} = -\frac{1}{q_1^2} + \frac{1}{q_2^2}\frac{dq_2}{dq_1} = -\frac{1}{q_1^2} + \frac{1}{q_2^2}\frac{f(q_1)}{f(q_2)}.$$

Set this equal to zero to find that the optimal choice must solve

$$\frac{q_2^2}{q_1^2} = \frac{f(q_1)}{f(q_2)}.$$

If the equal-tailed answer did this for all $\alpha \in (0\,, 1/2)$, then f would have to be symmetric; and it is not.

Chapter 11, Problem 30. **(a)** Since $\bar{X}_n$ is asymptotically $\mathrm{N}(\mu\,, \mu/n)$ and $g(\mu) = \sqrt{\mu}$, it follows that $c = \sqrt{\mu}$ and $g'(\mu) = 1/(2\sqrt{\mu})$. Therefore, $|cg'(\mu)|^2$ is a constant [namely, 1/4] and so g is a variance-stabilizing transformation. And by Theorem 7.7.6 (p. 249),

$$\sqrt{\bar{X}_n} \stackrel{d}{\approx} \mathrm{N}\left(\sqrt{\mu}\,,\, \frac{1}{4n}\right).$$

(b) We know that for n large,

$$P\left\{\left|\sqrt{\bar{X}_n} - \sqrt{\mu}\right| \le \frac{z_{1-(\alpha/2)}}{2\sqrt{n}}\right\} \approx 1 - \alpha.$$

Therefore, an approx. $100(1-\alpha)\%$ CI for $\sqrt{\mu}$ is

$$\left(\sqrt{\bar{X}_n} - \frac{z_{1-(\alpha/2)}}{2\sqrt{n}}\ ,\ \sqrt{\bar{X}_n} + \frac{z_{1-(\alpha/2)}}{2\sqrt{n}}\right),$$

and for μ, it translates to

$$\left(\left|\sqrt{\bar{X}_n} - \frac{z_{1-(\alpha/2)}}{2\sqrt{n}}\right|^2\ ,\ \left|\sqrt{\bar{X}_n} + \frac{z_{1-(\alpha/2)}}{2\sqrt{n}}\right|^2\right).$$

(c) By the central limit theorem,

$$\frac{\sqrt{n}\left[\bar{X}_n - \theta\right]}{\theta} \xrightarrow{d} \mathrm{N}(0\,,1).$$

So, $c(\theta) = \theta$, and we seek to find a function $g(\theta)$ such that $|c(\theta)g'(\theta)|^2 = \theta^2|g'(\theta)|^2$ is a constant. Set $g(\theta) := \ln\theta$ to see that indeed $\theta^2|g'(\theta)|^2 = 1$. Theorem 7.7.6 (p. 249) then tells us that

$$\sqrt{n}\left[\ln\bar{X}_n - \ln\theta\right] \xrightarrow{d} \mathrm{N}(0\,,1).$$

So an approximate $100(1-\alpha)\%$ CI for $\ln\theta$ is

$$\left(\ln\bar{X}_n - \frac{z_{1-(\alpha/2)}}{\sqrt{n}}\ ,\ \ln\bar{X}_n + \frac{z_{1-(\alpha/2)}}{\sqrt{n}}\right).$$

And an approximate $100(1-\alpha)\%$ CI for θ is

$$\left(\bar{X}_n \mathrm{e}^{-Q}\ ,\ \bar{X}_n \mathrm{e}^{Q}\right).$$

where $Q := z_{1-(\alpha/2)}/\sqrt{n}$.