Math 5090–001, Fall 2009

Solutions to Assignment 1

Chapter 11, Problem 1. (a) The $100(1-\alpha)\%$ confidence interval is

$$\bar{X} \pm z_{1-(\alpha/2)} \frac{\sigma}{\sqrt{n}} = 19.3 \pm z_{1-(\alpha/2)} \frac{3}{4}.$$

We want $1-\alpha = 0.9$ confidence interval; so $z_{1-(\alpha/2)} \approx 1.645$. So the confidence intervals is 19.3 ± 1.23375; that is, $(18.06625\,, 20.53375)$, approximately.

(b) The lower confidence interval for μ is

$$\bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}} \approx 19.3 - 1.285\frac{3}{4} = 18.33625.$$

The upper confidence interval for μ is

$$\bar{X} + z_\alpha \frac{\sigma}{\sqrt{n}} \approx 19.3 + 1.285\frac{3}{4} = 20.26375.$$

(c) The length of the confidence interval (11.2.7) is

$$\lambda = 2z_{1-(\alpha/2)} \frac{\sigma}{\sqrt{n}}.$$

Solve for n to obtain

$$n = \left(\frac{2z_{1-(\alpha/2)}\sigma}{\lambda} \right)^2 = \frac{4z^2_{1-(\alpha/2)}\sigma^2}{\lambda^2}.$$

If we want to ensure that $\lambda = 2$, then we need

$$n \approx \frac{4(1.645)^2 9}{4} = 24.354225.$$

Since n has to be an integer, it follows that the requisite n is 25 or more.

(d) We have the confidence interval [see (11.3.5)]

$$\bar{X} \pm t_{1-(\alpha/2)}(n-1)\frac{S}{\sqrt{n}} \approx 19.3 \pm 1.753\frac{\sqrt{10.24}}{4} = (17.8976\,, 20.7024)$$

(e) We apply (11.3.6) to obtain

$$\left(\frac{(n-1)S^2}{\chi^2_{1-(\alpha/2)}(n-1)}\,, \frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)}\right) \approx \left(\frac{15 \times 10.24}{32.8}\,, \frac{15 \times 10.24}{4.6}\right) \approx (4.683\,, 33.391).$$

Chapter 11, Problem 5. **(a)** First of all, recall that the density function of every X_j is

$$f(x\,,\eta) = \begin{cases} e^{-(x-\eta)} & \text{if } x > \eta, \\ 0 & \text{if } x \leq \eta. \end{cases}$$

Now we use a trick from Math 5080 to compute the density of $Q := X_{1:n} - \eta$. Namely, we first compute $F_Q(a)$, and then differentiate with respect to a. Here are the details: For all $a > 0$,

$$\begin{aligned} F_Q(a) &= 1 - P\{X_{1:n} - \eta > a\} \\ &= 1 - P\{X_1 > a + \eta\,, \cdots, X_n > a + \eta\} \\ &= 1 - (P\{X_1 > a + \eta\})^n . \end{aligned}$$

Because

$$P\{X_1 > a + \eta\} = \int_{a+\eta}^{\infty} e^{-(x-\eta)}\,dx = e^{-a},$$

it follows that

$$F_Q(a) = \begin{cases} 1 - e^{-na} & \text{if } a > 0 \\ 0 & \text{if } a \leq 0. \end{cases}$$

In particular, Q is distributed as EXP$(1/n)$, a distribution that does not depend on the parameter η.

(b) From part (a) we know that for all $a > 0$, $P\{Q \leq a\} = 1 - e^{na}$.

Let q_1 be the value of a that makes this probability $(1-\gamma)/2$; that is,

$$1 - e^{-nq_1} = \frac{1-\gamma}{2} \quad\Rightarrow\quad e^{-nq_1} = \frac{1+\gamma}{2} \quad\Rightarrow\quad q_1 = -\frac{1}{n}\log\left(\frac{1+\gamma}{2}\right).$$

Thus, $P\{Q \le q_1\} = (1-\gamma)/2$ [$=\alpha/2$]. This is not silly because it is possible to check that $q_1 > 0$ for $0 < \gamma < 1$.

Also let us find $q_2 > 0$ such that $P\{Q \ge q_2\} = (1-\gamma)/2$ as follows:

$$\frac{1-\gamma}{2} = P\{Q \ge q_2\} = 1 - F_Q(q_2) = e^{-nq_2}.$$

Consequently,

$$q_2 = -\frac{1}{n}\log\left(\frac{1-\gamma}{2}\right).$$

[This too is > 0.] Now, with q_1 and q_2 as above: $P\{q_1 \le Q \le q_2\} = \gamma$; but "$q_1 \le Q \le q_2$" is the same statement as "$q_1 \le X_{1:n} - \eta \le q_2$," which is in turn the same as "$X_{1:n} - q_2 \le \eta \le X_{1:n} - q_1$." That is, the confidence interval that we seek is

$$\left(X_{1:n} + \frac{1}{n}\log\left(\frac{1-\gamma}{2}\right),\, X_{1:n} + \frac{1}{n}\log\left(\frac{1+\gamma}{2}\right)\right).$$

(c) I am not sure what θ is in this problem. But it is irrelevant information, in any event. "A 90% confidence interval" implies that $\gamma = 0.9$, whence the CI is the following [thanks to part (b)]:

$$\left(162 + \frac{1}{19}\log\left(\frac{0.1}{2}\right),\, 162 + \frac{1}{19}\log\left(\frac{1.9}{2}\right)\right) \approx (161.84233\,, 161.9973)\,.$$

Chapter 11, Problem 8. **(a)** Note that

$$P\left\{X_{n:n} \le \theta \le 2X_{n:n}\right\} = P\left\{\frac{\theta}{2} \le X_{n:n} \le \theta\right\} = F_{X_{n:n}}(\theta) - F_{X_{n:n}}(\theta/2).$$

Because

$$F_{X_{n:n}}(a) = P\left\{X_1 \le a\,, \ldots, X_n \le a\right\} = (P\{X_1 \le a\})^n = \left(\frac{a}{\theta}\right)^n,$$

it follows that

$$P\{X_{n:n} \le \theta \le 2X_{n:n}\} = 1 - 2^{-n}.$$

(b) More generally, but still as in part (a), for all $c > 1$,

$$P\{X_{n:n} \le \theta \le cX_{n:n}\} = 1 - c^{-n}.$$

To find the correct c, set the preceding equal to $1 - \alpha$:

$$1 - c^{-n} = 1 - \alpha \;\Rightarrow\; c^{-n} = \alpha \;\Rightarrow\; c = \alpha^{-1/n}.$$

That is, $(X_{n:n}\,, \alpha^{-1/n}X_{n:n})$ is a $100(1-\alpha)\%$ confidence interval for θ.

Chapter 11, Problem 14. **(a)** Since $X \sim \mathrm{POI}(100\lambda)$, we want to apply Theorem 11.4.3. Note that

$$G(s\,,\lambda) = \sum_{j=0}^{s} e^{-100\lambda}\frac{(100\lambda)^j}{j!}.$$

Now $EX = 100\lambda$, and we have observed X to be 5. So a rough guess for λ is something like $\lambda \approx 0.05$. In other words, any reasonable statistical method is likely to produce a value of λ_U in the interval $I := (0\,,1)$. This is an important observation:

Suppose we could find an increasing function h_1 such that $G(h_1(\lambda)\,,\lambda) = 0.025$ for all λ close to zero. Then Theorem 11.4.3 tells us to solve for λ_U that solves $h_1(\lambda) = 5$, and that λ_U is a conservative upper 95% confidence limit for λ. In other words, if h_1 were decreasing, then we solve $G(5\,,\lambda_U) = 0.025$; i.e.,

$$0.025 = \sum_{j=0}^{5} e^{-100\lambda_U}\frac{(100\lambda_U)^j}{j!}$$

And this tells us (see Table 2, page 602 of your text) that

$$10 \le 100\lambda_U \le 15.$$

So certainly 15 is a conservative upper 95% confidence limit for the mean number 100λ of defects in this case.

This is the correct answer; it remains to verify the stated monotonicity of h_1. That, in turn, follows if we could show that $G(s\,,\lambda)$ is decreasing in λ for all s fixed (why?). This requires only a direct computation:

$$\begin{aligned}
\frac{d}{d\lambda}G(s\,,\lambda) &= \sum_{j=0}^{s} e^{-100\lambda}\frac{(100\lambda)^j}{j!}\left(\frac{j}{\lambda}-100\right)\\
&= e^{-100\lambda}\left(\sum_{j=0}^{s}\frac{(100\lambda)^j}{j!}\frac{j}{\lambda} - 100\sum_{j=0}^{s}\frac{(100\lambda)^j}{j!}\right)\\
&= e^{-100\lambda}\left(\sum_{j=1}^{s}\frac{(100\lambda)^j}{j!}\frac{j}{\lambda} - 100\sum_{j=0}^{s}\frac{(100\lambda)^j}{j!}\right)\\
&= e^{-100\lambda}\left(\sum_{j=1}^{s}\frac{(100\lambda)^j}{(j-1)!}\frac{1}{\lambda} - 100\sum_{j=0}^{s}\frac{(100\lambda)^j}{j!}\right)\\
&= e^{-100\lambda}\left(100\lambda\sum_{j=1}^{s}\frac{(100\lambda)^{j-1}}{(j-1)!}\frac{1}{\lambda} - 100\sum_{j=0}^{s}\frac{(100\lambda)^j}{j!}\right)\\
&= -e^{-100\lambda}\frac{(100\lambda)^s}{s!} \leq 0.
\end{aligned}$$

(b) This is as in (a), but we want a CI for λ and not 100λ this time; also, $X = 15$ is the observed value. That is, find λ_U such that

$$0.025 = \sum_{j=0}^{15} e^{-100\lambda_U}\frac{(100\lambda_U)^j}{j!}.$$

And Table 2 yields only that $\lambda_U \gg 15$.