Math 5090–001, Fall 2009
Solutions to Assignment 10

Chapter 15, Problem 25. We know:

$$\hat{\sigma}^2 = \frac{(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})}{n} \qquad \left[\text{which is } = \frac{1}{n}\|\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}\|^2\right].$$

Since $(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})' = \boldsymbol{Y}' - (\boldsymbol{X}\hat{\boldsymbol{\beta}})'$, we can write the numerator of $\hat{\sigma}^2$ as

$$\begin{aligned}
\boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}} - (\boldsymbol{X}\hat{\boldsymbol{\beta}})'\boldsymbol{Y} + (\boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{X}\hat{\boldsymbol{\beta}}) &= \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{Y} + \hat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}} \\
&= \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}} - (\boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}})' + \hat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}} \\
&= \boldsymbol{Y}'\boldsymbol{Y} - 2\boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}}
\end{aligned}$$

[The last line holds because all of these terms are scalars. In particular, $\boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}}$ is a scalar. So it is equal to its own transpose.] Plug in $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$ in this to obtain:

$$\begin{aligned}
\boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}} &= \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}; \qquad \text{and} \\
\hat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}} &= \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \\
&= \boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}}.
\end{aligned}$$

Therefore, the numerator of $\hat{\sigma}^2$ is $\boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{Y}'(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})$. Divide this by n to finish.