Math 5080–1
Solutions to homework 8

July 26, 2012

 # 17 (a) The PDF in question is

$$f(x) = e^{-(x-\eta)} I\{x > \eta\}.$$

This is a range-dependent exponential class with $q_1(\eta) = \eta$, $q_2(\eta) = \infty$, $c(\eta) = e^{\eta}$, $h(x) = e^{-x}$. Part 3 of Theorem 10.4.5 (p. 350) tells us that $X_{1:n}$ is complete sufficient for η.

17 (b) Recall from the solution to Exercise 4 that $E(X_{1:n}) = \eta + (1/n)$. Therefore, $X_{1:n} - (1/n)$ is an unbiased estimator that is based on a complete sufficient statistic. Thanks to Lehmann-Scheffé, it is UMVUE for η.

17 (c) Let us first find an exact expression for the pth percentile x_p in terms of η: We want x_p to solve

$$p = P\{X_1 < x_p\} = \int_{\eta}^{x_p} e^{-(x-\eta)}\, dx = 1 - e^{-(x_p - \eta)}.$$

That is,

$$x_p = \eta - \ln(1-p).$$

For the UMVUE $\hat{x}_p$ of x_p, we look at a function S of $X_{1:n}$ that is unbiased [i.e., we want $E(\hat{x}_p) = \eta - \ln(1-p)$]. By the result of (b),

$$\hat{x}_p := X_{1:n} - \frac{1}{n} - \ln(1-p)$$

does the job.

 # 19. The problem is that if $\mu = 0$, then $N(0\,,0)$ has to be zero with probability one. Therefore, Θ has to be defined as the collection of all μ such that $\mu \neq 0$; i.e., $\Theta = (-\infty\,,0) \cup (0\,,\infty)$. This does not have the right form [it is not an interval].

 # 21(a) Here,

$$f(x\,,p) = p^x (1-p)^{1-x} I\{x = 0\,,1\}.$$

We can write this as

$$f(x\,,p) = (1-p)\left(\frac{p}{1-p}\right)^x I\{x=0\,,1\} = (1-p)e^{q_1(p)t_1(x)}I\{x=0\,,1\},$$

where $q_1(p) = \ln(p/(1-p))$ and $t_1(x) = x$. This is a REC. Therefore, Theorem 10.4.2 implies that $\sum_{i=1}^n t_1(X_i) = \sum_{i=1}^n X_i$ is complete and sufficient for p. We know from general theory that if

$$S := \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^n X_i^2 - \frac{n}{n-1}(\bar{X})^2,$$

then $E(S) = p(1-p)$. Because $X_i = 0$ or 1, $\sum_{i=1}^n X_i^2$, for this particular example. Therefore,

$$S = \frac{1}{n-1}\sum_{i=1}^n X_i - \frac{n}{n-1}(\bar{X})^2 = \frac{n}{n-1}\bar{X}(1-\bar{X})$$

is both unbiased and a function of the complete sufficient statistic. Therefore, S is UMVUE for $p(1-p)$.

21 (b) We may use S from part (a), and recall that $E(S) = p(1-p) = p - p^2 = E(\bar{X}) - p^2$. Therefore, if we define

$$\tilde{S} := \bar{X} - S,$$

then $E\tilde{S} = p - p(1-p) = p^2$. Since $\tilde{S}$ is a function of $\sum_{i=1}^n X_i$, it is UMVUE for p^2. We can simplify $\tilde{S}$ further as follows:

$$\tilde{S} = \bar{X} - \frac{n}{n-1}\bar{X}(1-\bar{X}) = \bar{X}\left[1 - \frac{1-\bar{X}}{n-1}\right] = \bar{X}\cdot\frac{n+\bar{X}}{n-1}.$$