Math 5080–1
Solutions to homework 7

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 # 2. Here, $X_1, \ldots, X_n$ are i.i.d. GEO(p); therefore, the joint mass function is

$$f(x_1, \ldots, x_n) = p^n (1-p)^{x_1+\cdots+x_n-n} \cdot I\{x_1, \ldots, x_n = 1, 2, \ldots\}.$$

Also, $S_n = X_1 + \cdots + X_n$ is negative binomial NB$(n\,, p)$ [see the MGFs in your text's cover, for example]. That is,

$$f_S(s) = \binom{s-1}{n-1} p^n (1-p)^{s-n} \cdot I\{s = n, n+1, \ldots,\}.$$

Therefore, (10.21) tells us that

$$\begin{aligned}
f_{\boldsymbol{X}|S}(x_1\,, \ldots, x_n \,|\, s) &= \frac{f(x_1\,, \ldots, x_n)}{f_S(s)} \cdot I\{x_1 + \cdots + x_n = s\} \\
&= \frac{p^n(1-p)^{x_1+\cdots+x_n-n}}{\binom{s-1}{n-1} p^n (1-p)^{s-n}} \cdot I\{x_1 + \cdots + x_n = s\} \\
&= \frac{p^n(1-p)^{s-n}}{\binom{s-1}{n-1} p^n (1-p)^{s-n}} \cdot I\{x_1 + \cdots + x_n = s\} \\
&= \binom{s-1}{n-1}^{-1} \cdot I\{x_1 + \cdots + x_n = s\}.
\end{aligned}$$

This quantity does not depend on p; therefore, S is a sufficient statistic for p.

 # 3. Here,

$$f(x_1\,, \ldots, x_n) = (2\pi\theta)^{-n/2} \exp\left(-\frac{1}{2\theta} \sum_{j=1}^n x_j^2\right),$$

and $S/\theta \sim \chi^2(n)$. In particular,

$$P\{S \leq s\} = P\left\{\chi^2(n) \leq \frac{s}{\theta}\right\}.$$

Differentiate both sides (d/ds) to see that

$$\begin{aligned} f_S(s) &= f_{\chi^2(n)}(s/\theta) \times \frac{1}{\theta} \\ &= \frac{1}{2^{n/2}\Gamma(n/2)} \left(\frac{s}{\theta}\right)^{(n/2)-1} e^{-s/(2\theta)} \times \frac{1}{\theta}. \end{aligned}$$

It follows from (10.21) that

$$\begin{aligned} f_{\boldsymbol{X}|S}(x_1, \ldots, x_n \,|\, s) &= \frac{f(x_1, \ldots, x_n)}{f_S(s)} \cdot I\left\{x_1^2 + \cdots + x_n^2 = s\right\} \\ &= \frac{(2\pi\theta)^{-n/2} \exp\left(-\frac{1}{2\theta}\sum_{j=1}^n x_j^2\right)}{\frac{1}{2^{n/2}\Gamma(n/2)} \left(\frac{s}{\theta}\right)^{(n/2)-1} e^{-s/(2\theta)} \times \frac{1}{\theta}} \cdot I\left\{x_1^2 + \cdots + x_n^2 = s\right\} \\ &= \frac{\Gamma(n/2)}{\pi^{n/2} s^{(n/2)-1}} \cdot I\left\{x_1^2 + \cdots + x_n^2 = s\right\}. \end{aligned}$$

This does not depend on θ; therefore, S is sufficient for θ.

 # 4. First of all,

$$f(x_1, \ldots, x_n) = e^{-\sum_{j=1}^n (x_j - \eta)} \cdot I\left\{x_{1:n} > \eta\right\}.$$

Now we work on the distribution of $S = X_{1:n}$. Since

$$P\left\{S \le s\right\} = 1 - \left(P\{X_1 > s\}\right)^n = 1 - \left(1 - F(s)\right)^n \quad \text{for all } s,$$

we differentiate both sides (d/ds) to see that

$$f_S(s) = n\left(1 - F(s)\right)^{n-1} f(s).$$

Since $f(x) = e^{-(x-\eta)} \cdot I\{x > \eta\}$,

$$F(s) = \int_\eta^s e^{-(a-\eta)}\, da = 1 - e^{-(s-\eta)} \quad \text{if } s > \eta.$$

If $s \le \eta$ then $F(s) = 0$. Therefore,

$$f_S(s) = ne^{-n(s-\eta)} \cdot I\{s > \eta\} \Rightarrow S \sim \text{DE}(n, \eta)$$

And by (10.21),

$$\begin{aligned} f_{\boldsymbol{X}|S}(x_1, \ldots, x_n \,|\, s) &= \frac{f(x_1, \ldots, x_n)}{f_S(s)} \cdot I\left\{x_{1:n} = s\right\} \\ &= \frac{e^{-\sum_{j=1}^n (x_j - \eta)} \cdot I\left\{x_{1:n} > \eta\right\}}{ne^{-n(s-\eta)} \cdot I\{s > \eta\}} \cdot I\left\{x_{1:n} = s\right\} \\ &= \frac{e^{-\sum_{j=1}^n (x_j - s)}}{n} \cdot I\left\{x_{1:n} = s\right\}. \end{aligned}$$

This quantity does not depend on η.

 # 6. The joint mass function of our data is

$$f(x_1, \ldots, x_n) = \prod_{i=1}^n \binom{m_i}{x_i} p^{x_i} q^{m_i - x_i} \cdot I\{x_i = 0, \ldots, m_i \text{ for all } i = 1, \ldots, n\}$$
$$= C p^{\sum_{i=1}^n x_i} q^{\sum_{i=1}^n (m_i - x_i)} \cdot I\{x_i = 0, \ldots, m_i \text{ for all } i = 1, \ldots, n\},$$

where $C := \prod_{i=1}^n \binom{m_i}{x_i}$. That is,

$$f(x_1, \ldots, x_n) = C'(p/q)^{\sum_{i=1}^n x_i} \cdot I\{x_i = 0, \ldots, m_i \text{ for all } i = 1, \ldots, n\},$$

where $C' := C q^{\sum_{i=1}^n m_i}$. The factorization theorem does the rest for us.

 # 7. Let us apply the factorization theorem. The joint mass function is

$$f(x_1, \ldots, x_n) = \prod_{i=1}^n \binom{x_i - 1}{r_i - 1} p^{r_i} q^{x_i - r_i} \cdot I\{x_i = r_1, r_1 + 1, \ldots \text{ for all } i \le n\}$$
$$= C q^{\sum_{i=1}^n x_i} \cdot I\{x_i = r_1, r_1 + 1, \ldots \text{ for all } i \le n\},$$

where $C := \prod_{i=1}^n \binom{x_i - 1}{r_i - 1}(p/q)^{r_i}$. Therefore, $S := \sum_{i=1}^n X_i$ is sufficient for p.

 # 16. The likelihood function is

$$L(p) = \prod_{i=1}^n \binom{X_i - 1}{r_i - 1} p^{r_i} q^{X_i - r_i},$$

therefore, the log-likelihood function is

$$\ln L(p) = \sum_{i=1}^n \log \binom{X_i - 1}{r_i - 1} + \ln p \sum_{i=1}^n r_i + \ln(1 - p) \sum_{i=1}^n (X_i - r_i).$$

Therefore,

$$\frac{d}{dp} \ln L(p) = \frac{\sum_{i=1}^n r_i}{p} - \frac{\sum_{i=1}^n (X_i - r_i)}{1 - p}.$$

Set this equal to zero in order to see that the usual MLE of p is

$$\hat{p} = \frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n (X_i - r_i) + \sum_{i=1}^n r_i} = \frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n X_i}.$$

Now let us maximize the mass function of $S := \sum_{i=1}^n X_i$. Examine the MGF's to see that $S \sim \mathrm{NB}(\sum_{i=1}^n r_i, p)$. Therefore,

$$f_S(s) = \binom{s - 1}{\sum_{i=1}^n r_i - 1} p^{\sum_{i=1}^n r_i} q^{s - \sum_{i=1}^n r_i} \cdot I\left\{s \ge \sum_{i=1}^n r_i\right\}.$$

The corresponding likelihood function

$$L_S(p) = \binom{S-1}{\sum_{i=1}^n r_i - 1} p^{\sum_{i=1}^n r_i} q^{S-\sum_{i=1}^n r_i},$$

whose log is

$$\ln L_S(p) = \ln \binom{S-1}{\sum_{i=1}^n r_i - 1} + \ln p \sum_{i=1}^n r_i + \left(S - \sum_{i=1}^n r_i\right) \ln(1-p).$$

This yields

$$\frac{d}{dp} \ln L_S(p) = \frac{\sum_{i=1}^n r_i}{p} - \frac{S - \sum_{i=1}^n r_i}{1-p}.$$

Set the preceding to zero to see that $\hat{p} = \sum_{i=1}^n r_i / S$, as before. So the answer is "yes," the two procedures yield the same MLE.