Math 5080–1
Solutions to homework 6

July 6, 2012

 # 22 (a). The likelihood function is

$$L(\mu) = (18\pi)^{-n/2} \exp\left(-\frac{1}{18}\sum_{j=1}^n (X_j - \mu)^2\right).$$

Therefore,

$$\ln L(\mu) = -\frac{n}{2}\log(18\pi) - \frac{1}{18}\sum_{j=1}^n (X_j - \mu)^2.$$

It derivative is

$$\frac{\partial}{\partial\mu}\ln L(\mu) = \frac{1}{9}\sum_{j=1}^n (X_j - \mu) = \frac{n}{9}\bar{X} - \frac{n\mu}{9}.$$

Therefore, if T is any unbiased estimator of μ, then

$$\text{Var}(T) \geq \frac{1}{\text{Var}\left(\frac{\partial}{\partial\mu}\ln L(\mu)\right)} = \frac{1}{(n^2/81)\text{Var}(\bar{X})} = \frac{1}{(n^2/81)\times(9/n)} = \frac{9}{n}.$$

 # 22 (b). Yes. It is unbiased and achieves the CRLB.

 # 22 (c). $x_{0.95}$ is the 95th percentile if and only if

$$0.95 = P\{X_1 \leq x_{0.95}\} = \Phi\left(\frac{x_{0.95}-\mu}{3}\right) \Rightarrow \frac{x_{0.95}-\mu}{3} = 1.645 \Rightarrow x_{0.95} = 4.935+\mu.$$

By the MLE principle, the MLE of $x_{0.95}$ is

$$\hat{x}_{0.95} = 4.935 + \bar{X} \Rightarrow E\hat{x}_{0.95} = 4.935 + \mu = x_{0.95}.$$

Moreover, basic properties of variances ensure that

$$\text{Var}\hat{x}_{0.95} = \text{Var}\left(4.935 + \bar{X}\right) = \text{Var}(\bar{X}) = \frac{9}{n} = \text{CRLB!}$$

Therefore, the MLE of $x_{0.95}$ is UMVUE.

 # 24 (a). We use the MGF table of the back cover to see that

$$E\hat{\theta} = Ee^{-X} = M_X(-1) = \exp\left(\mu(e^{-1}-1)\right) = \theta^{e^{-1}-1} \neq \theta.$$

Therefore, $\hat{\theta}$ is biased.

 # 24 (b). We observe that $\tilde{\theta} = u(X)$ can also be written as $I\{X=0\}$. Therefore, $E\tilde{\theta} = P\{X=0\} = e^{-\mu}$.

 # 24 (c). We have

$$\begin{aligned}
\mathrm{Var}(\hat{\theta}) &= E\left(e^{-2X}\right) - \left[E\left(e^{-X}\right)\right]^2 = M_X(-2) - [M_X(-1)]^2 \\
&= \exp\left(\mu\left(e^{-2}-1\right)\right) - \exp\left(2\mu\left(e^{-1}-1\right)\right) \\
&= e^{-2\mu}\left[e^{\mu(e^{-2}+1)} - e^{2\mu/e}\right],
\end{aligned}$$

and

$$\mathrm{Bias}(\hat{\theta}) = E\hat{\theta} - \theta = \exp\left(\mu\left(e^{-1}-1\right)\right) - e^{-\mu} = e^{-\mu}\left[e^{\mu/e}-1\right].$$

Therefore,

$$\begin{aligned}
\mathrm{MSE}(\hat{\theta}) &= \mathrm{Bias}^2 + \mathrm{Var} \\
&= e^{-2\mu}\left[e^{\mu/e}-1\right]^2 + e^{-2\mu}\left[e^{\mu(e^{-2}+1)} - e^{2\mu/e}\right] \\
&= e^{-2\mu}\left[e^{\mu(e^{-2}+1)} + 1 - 2e^{\mu/e}\right]
\end{aligned}$$

Also, because $(\tilde{\theta})^2 = \tilde{\theta}$,

$$\mathrm{MSE}(\tilde{\theta}) = \mathrm{Var}(\tilde{\theta}) = E\tilde{\theta} - \left(E\tilde{\theta}\right)^2 = e^{-\mu} - e^{-2\mu} = e^{-2\mu}\left(e^{\mu}-1\right).$$

The ratio is

$$\mathcal{R}(\mu) := \frac{\mathrm{MSE}(\hat{\theta})}{\mathrm{MSE}(\tilde{\theta})} = \frac{e^{\mu(e^{-2}+1)} + 1 - 2e^{\mu/e}}{e^{\mu}-1}.$$

In particular,

$$\mathcal{R}(1) = \frac{e^{e^{-2}+1} + 1 - 2e^{1/e}}{e-1} \approx 0.7117 < 1,$$

and

$$\mathcal{R}(2) = \frac{e^{2(e^{-2}+1)} + 1 - 2e^{2/e}}{e^2-1} \approx 1.0912 > 1.$$

Therefore, $\hat{\theta}$ is better than $\tilde{\theta}$ when $\mu = 1$, and worse when $\mu = 2$.

 # 28 (a). First,

$$\mathrm{Var}(\hat{\theta}_1) = \mathrm{Var}(\bar{X}) = \frac{\mathrm{Var}(X_1)}{n} = \frac{\theta^2}{n}.$$

Next,

$$\mathrm{Var}(\hat{\theta}_2) = \left(\frac{n}{n+1}\right)^2 \cdot \mathrm{Var}(\bar{X}) = \left(\frac{n}{n+1}\right)^2 \cdot \frac{\mathrm{Var}(X_1)}{n} = \frac{n\theta^2}{(n+1)^2}.$$

 # 28 (b). $E(\hat{\theta}_1) = E(\bar{X}) = \theta$ and $E(\hat{\theta}_2) = n\theta/(n+1)$. Therefore, $\hat{\theta}_1$ is unbiased, and

$$\mathrm{Bias}(\hat{\theta}_2) = \frac{n\theta}{n+1} - \theta = -\frac{\theta}{n+1}.$$

Consequently,

$$\mathrm{MSE}(\hat{\theta}_1) = \frac{\theta^2}{n}, \quad \mathrm{MSE}(\hat{\theta}_2) = \frac{n\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} = \frac{\theta^2}{n+1}.$$

 # 28 (c). Because $n/(n+1) < 1$, $\mathrm{Var}(\hat{\theta}_1)$ is always greater than $\mathrm{Var}(\hat{\theta}_2)$. There is no need to consider special values of n.

 # 28 (d). No need to consider special values of n: $\hat{\theta}_2$ is always better than $\hat{\theta}_1$, though it is biased.

 # 28 (e). Not covered.

 # 34 (a). The likelihood function is

$$L(p) = p^n (1-p)^{n\bar{X}} \Rightarrow \ln L(p) = n \ln p + n\bar{X} \log(1-p).$$

Differentiate:

$$\frac{\partial}{\partial p} \ln L(p) = \frac{n}{p} - \frac{n\bar{X}}{1-p},$$

and set it equal to zero to see that $\hat{p}$ must solve

$$\frac{1}{\hat{p}} = \frac{\bar{X}}{1-\hat{p}} \Rightarrow \hat{p} = \frac{1}{1+\bar{X}}.$$

 # 34 (b). We apply the MLE principle:

$$\hat{\theta} = \frac{1-\hat{p}}{\hat{p}} = \frac{1-(1+\bar{X})^{-1}}{(1+\bar{X})^{-1}} = \bar{X}.$$

 # 34 (c). Note that $\theta = \tau(p)$, where $\tau(p) = (1-p)/p$. In particular,

$$\tau'(p) = -1/p^2.$$

Also,

$$\operatorname{Var}\left(\frac{\partial}{\partial p}\ln L(p)\right) = \left(\frac{n}{1-p}\right)^2 \operatorname{Var}(\bar{X}) = \frac{n\operatorname{Var}(X_1)}{(1-p)^2} = \frac{n}{p^2(1-p)},$$

since the variance of a GEOM(p) is $(1-p)/p^2$. Therefore, the CRLB is

$$\frac{[\tau'(p)]^2}{\frac{n}{p^2(1-p)}} = \frac{p^{-4}}{\frac{n}{p^2(1-p)}} = \frac{(1-p)}{np^2}.$$

 # 34 (d). No. The MLE is biased, as can be seen from the following calculation:

$$E\hat{\theta} = E(\bar{X}) = E(X_1) = \frac{1}{p} \neq \frac{1-p}{p} \quad \text{for all } 0 < p < 1.$$

 # 34 (e). No. Its bias does not converge to zero as $n \to \infty$. Indeed,

$$\operatorname{Bias}(\hat{\theta}) = \frac{1}{p} - \frac{1-p}{p} = 1,$$

no matter the value of p!

 # 34 (f). $E(\bar{X}) = 1/p$, as we saw earlier. Also, $\operatorname{Var}(\bar{X}) = \operatorname{Var}(X_1)/n = (1-p)/(p^2 n)$. Therefore, by the CLT,

$$\frac{\bar{X} - (1/p)}{\sqrt{(1-p)/(p^2 n)}} \xrightarrow{d} \mathrm{N}(0\,, 1).$$

Equivalently,

$$\sqrt{n}\left(\bar{X} - \frac{1}{p}\right) \xrightarrow{d} \mathrm{N}\left(0\,, \frac{1-p}{p^2}\right)$$