Math 5080–1
Solutions to homework 5

June 28, 2012

 # 2. (a) Since $E(X_1) = 3/p$, equivalently $p = 3/E(X_1)$, a MME is $\hat{p} = 3/\bar{X}$

(b) Since $E(X_1) = 2\kappa$, a MME is $\hat{\kappa} = \bar{X}/2$.

(c) Since $E(X_1) = \theta\Gamma(3) = 2\theta$, a MME is $\hat{\theta} = \bar{X}/2$.

(d) Since $E(X_1) = \eta$, a MME is $\hat{\eta} = \bar{X}$. Also, $\mathrm{Var}(X_1) = 2\theta^2$, and therefore, $E(X_1^2) = 2\theta^2 + \eta^2$. Therefore, a MME for $2\theta^2 + \eta^2$ is $n^{-1}\sum_{j=1}^n X_j^2$, and hence a MME for $2\theta^2$ is $n^{-1}\sum_{j=1}^n X_j^2 - (\bar{X})^2 = n^{-1}\sum_{j=1}^n (X_j - \bar{X})^2$. Finally, this means that a MME for θ is

$$\hat{\theta} = \left(\frac{1}{2n}\sum_{j=1}^n (X_j - \bar{X})^2\right)^{1/2}.$$

(e) An MME for $E(X_1) = \eta - \gamma\theta$ is $\bar{X}$. An MME for $\mathrm{Var}(X_1) = (\pi^2\theta^2/6)$ is $n^{-1}\sum_{j=1}^n (X_j - \bar{X})^2$. Therefore, an MME for θ is

$$\hat{\theta} = \left(\frac{6}{\pi^2 n}\sum_{j=1}^n (X_j - \bar{X})^2\right)^{1/2}.$$

And therefore, an MME for η is

$$\hat{\eta} = \bar{X} + \gamma\hat{\theta} = \bar{X} + \gamma\left(\frac{6}{\pi^2 n}\sum_{j=1}^n (X_j - \bar{X})^2\right)^{1/2}.$$

(f) $\bar{X}$ is an MME for $E(X_1) = \theta/(\kappa - 1)$; $n^{-1}\sum_{j=1}^n (X_j - \bar{X})^2$ is an MME for $\theta^2\kappa/(k-2)(\kappa-1)^2$. Square $\bar{X}$ and then divide to see that

$$\frac{1}{n(\bar{X})^2}\sum_{j=1}^n (X_j - \bar{X})^2 \text{ is an MME for } \frac{\kappa}{\kappa - 2}.$$

I.e.,

$$\frac{\hat{\kappa}}{\hat{\kappa}-2} = \frac{1}{n(\bar{X})^2}\sum_{j=1}^{n}(X_j-\bar{X})^2 := \frac{\hat{\sigma}^2}{(\bar{X})^2},$$

where $\hat{\sigma}^2 := n^{-1}\sum_{j=1}^{n}(X_j-\bar{X})^2$. We solve to obtain the following:

$$\hat{\kappa}\left(\frac{\hat{\sigma}^2}{(\bar{X})^2}-1\right) = 2\frac{\hat{\sigma}^2}{(\bar{X})^2} \Rightarrow \hat{\kappa}\left(\hat{\sigma}^2-(\bar{X})^2\right) = 2\hat{\sigma}^2.$$

that is,

$$\hat{\kappa} = \frac{2\hat{\sigma}^2}{\hat{\sigma}^2-(\bar{X})^2}.$$

Finally, $\bar{X} = \hat{\theta}/(\hat{\kappa}-1)$; so that

$$\hat{\theta} = (\hat{\kappa}-1)\bar{X} = \frac{\hat{\sigma}^2+(\bar{X})^2)\bar{X}}{\hat{\sigma}^2-(\bar{X})^2}.$$

 # 4.

(a) $f(x) = p^x(1-p)^{1-x}\cdot I\{x=0,1\}$. Therefore, $L(p) = \prod_{j=1}^{n} p^{X_j}(1-p)^{1-X_j} = p^{n\bar{X}}(1-p)^{n(1-\bar{X})}$. Here, log likelihood is easier to work with. Indeed, $\log L(p) = n\bar{X}\log p + n(1-\bar{X})\log(1-p)$. Now

$$\frac{d}{dp}\log L(p) = \frac{n\bar{X}}{p} - \frac{n(1-\bar{X})}{1-p}.$$

Set this expression equal to zero: $p(1-\bar{X}) = (1-p)\bar{X} \Rightarrow \hat{p} = \bar{X}$.

(b) Here, $f(x) = p(1-p)^{x-1}\cdot I\{x=0,1,\ldots\}$. Therefore, $L(p) = p^n(1-p)^{n(\bar{X}-1)}$, and hence $\log L(p) = n\log p + n(\bar{X}-1)\log(1-p)$. Now

$$\frac{d}{dp}\log L(p) = \frac{n}{p} - \frac{n(\bar{X}-1)}{1-p}.$$

Set this expression equal to zero: $1-p = p(\bar{X}-1) \Rightarrow \hat{p} = 1/\bar{X}$.

(c) Here, $f(x) = \binom{x-1}{2}p^3(1-p)^{x-3}\cdot I\{x=3,4,\ldots\}$. Therefore,

$$L(p) = \prod_{j=1}^{n}\binom{X_j-1}{2}\cdot p^{3n}(1-p)^{n\bar{X}-3n},$$

$$\log L(p) = \sum_{j=1}^{n}\log\binom{X_j-1}{2} + 3n\log p + (n\bar{X}-3n)\log(1-p).$$

Now

$$\frac{d}{dp}\log L(p) = \frac{3n}{p} - \frac{n\bar{X}-3n}{1-p}.$$

Set this expression equal to zero: $3(1-p) = (\bar{X}-3)p \Rightarrow \hat{p} = 3/\bar{X}$.

(d) Here, $f(x) = (2\pi\theta)^{-1/2} \exp(-x^2/(2\theta))$. Therefore,

$$L(\theta) = (2\pi\theta)^{-n/2} \exp\left(-\frac{1}{2\theta}\sum_{j=1}^{n} X_j^2\right) \Rightarrow \log L(\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{j=1}^{n} X_j^2.$$

Now

$$\frac{d}{d\theta}\log L(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{j=1}^{n} X_j^2.$$

Set this expression equal to zero: $\hat{\theta} = n^{-1}\sum_{j=1}^{n} X_j^2$.

(e) Here, $f(x) = \theta^{-2} x \exp(-x/\theta) \cdot I\{x > 0\}$. Therefore, $L(\theta) = \theta^{-2n}\prod_{j=1}^{n} X_j \cdot \exp(-\sum_{j=1}^{n} X_j/\theta)$, whence $\log L(\theta) = -2n\log\theta + \sum_{j=1}^{n}\log X_j - \theta^{-1}\sum_{j=1}^{n} X_j$. Now

$$\frac{d}{d\theta}\log L(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{j=1}^{n} X_j.$$

Set this expression equal to zero: $\hat{\theta} = (2n)^{-1}\sum_{j=1}^{n} X_j$.

(f) Here, $f(x) = (2\theta)^{-1}\exp(-|x|/\theta)$. Therefore, $L(\theta) = (2\theta)^{-n}\exp(-\sum_{j=1}^{n}|X_j|/\theta)$, whence $\log L(\theta) = -n\log 2 - n\log\theta - \theta^{-1}\sum_{j=1}^{n}|X_j|$. Now

$$\frac{d}{d\theta}\log L(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{j=1}^{n}|X_j|.$$

Set this expression equal to zero: $\hat{\theta} = n^{-1}\sum_{j=1}^{n}|X_j|$.

(g) Here, $f(x) = \frac{1}{2}\theta^{-1/2}\exp(-x^{1/2}\theta^{-1/2}) \cdot I\{x > 0\}$. Therefore, $L(\theta) = 2^{-n}\theta^{-n/2}\exp(-\sum_{j=1}^{n} X_j^{1/2}\theta^{-1/2})$, whence $\log l(\theta) = -n\log 2 - (n/2)\log\theta - \theta^{-1/2}\sum_{j=1}^{n} X_j^{1/2}$. Now

$$\frac{d}{d\theta}\log L(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^{3/2}}\sum_{j=1}^{n} X_j^{1/2}.$$

Set this expression equal to zero:

$$\hat{\theta} = \left(\frac{1}{n}\sum_{j=1}^{n} X_j^{1/2}\right)^2.$$

(h) Here, $f(x) = \kappa(1+x)^{-\kappa-1} \cdot I\{x > 0\}$. Therefore, $L(\kappa) = \kappa^n\prod_{j=1}^{n}(1 + X_j)^{-\kappa-1}$, whence $\log L(\kappa) = n\log\kappa - (\kappa+1)\sum_{j=1}^{n}\log(1+X_j)$. Now

$$\frac{d}{d\kappa}\log L(\kappa) = \frac{n}{\kappa} - \sum_{j=1}^{n}\log(1+X_j) \Rightarrow \hat{\kappa} = \left(\frac{1}{n}\sum_{j=1}^{n}\log(1+X_j)\right)^{-1}.$$

 # 6 (a). First of all, note that

$$L(\theta_1\,,\theta_2) = (\theta_2 - \theta_1)^{-n} \cdot I\left\{\theta_1 \le \min_{1\le j\le n} X_j\ ,\ \ \max_{1\le j\le n} X_j \le \theta_2\right\}.$$

This is maximized by minimizing $\theta_2 - \theta_1$ over the range where $I\{\cdots\} = 1$; i.e.,

$$\hat\theta_1 = \min_{1\le j\le n} X_j, \quad \hat\theta_2 = \max_{1\le j\le n} X_j.$$

#6. (b) Once again we compute the likelihood function:

$$L(\theta\,,\eta) = \theta^n \eta^{n\theta} \prod_{j=1}^n X_j^{-\theta-1} \cdot I\{\min_{1\le j\le n} X_j \ge \eta\}.$$

The preceding [without the indicator] is an increasing function of η. Therefore, it is maximized at $\hat\eta := \min_{1\le j\le n} X_j$. We can now apply the likelihood principle and a little calculus in order to compute $\hat\theta$:

$$L(\theta\,,\hat\eta) = \theta^n \left(\min_{1\le j\le n} X_j\right)^{n\theta} \prod_{j=1}^n X_j^{-\theta-1} \Rightarrow \log L(\theta\,,\hat\eta) = n\log\theta + n\theta\log\min_{1\le j\le n} X_j - (\theta+1)\sum_{j=1}^n \log X_j.$$

Now

$$\frac{d}{d\theta}\log L(\theta\,,\hat\eta) = \frac{n}{\theta} + n\log\min_{1\le j\le n} X_j - \sum_{j=1}^n \log X_j.$$

Set this equal to zero to find that

$$\hat\theta = n\left(\sum_{j=1}^n \log X_j - n\log\min_{1\le j\le n} X_j\right)^{-1}.$$

 # 8. (a) We know that

$$P\{X > c\} = 1 - \Phi\left(\frac{c-\mu}{\sigma}\right).$$

Therefore, according to the likelihood principle, the MLE is

$$1 - \Phi\left(\frac{c-\hat\mu}{\hat\sigma}\right) = 1 - \Phi\left(\frac{c - \bar X}{\sqrt{n^{-1}\sum_{j=1}^n (X_j - \bar X)^2}}\right).$$

#8. (b) The 95th percentile of X is the point x such that

$$0.95 = P\{X \le x\} = \Phi\left(\frac{x-\mu}{\sigma}\right) \Rightarrow \frac{x-\mu}{\sigma} \approx 1.645;$$

see Table 3. In other words, $x \approx 1.645\sigma + \mu$. Therefore, we apply the likelihood principle to see that $\hat{x} \approx 1.645\sqrt{n^{-1}\sum_{j=1}^n (X_j - \bar{X})^2} + \bar{X}$.

 # 10. (a) Since $f(x) = \frac{1}{2}\exp(-|x-\eta|)$,

$$L(\eta) = 2^{-n}\exp\left(-\sum_{j=1}^n |X_j - \eta|\right) \Rightarrow \log L(\eta) = -n\log 2 - \sum_{j=1}^n |X_j - \eta|.$$

Therefore, $\hat{\eta}$ minimizes $H(\eta)$, where H denotes the function

$$H(a) := \sum_{j=1}^n |X_j - a|.$$

Claim: The minimum of $H(a)$ occurs at $a =$ the median of the sample $X_1, \ldots, X_n$; we denote the sample median by med.

> In order to verify the Claim, let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the order statistics. [The inequalities are strict since the data has a pdf, so no two variables are ever equal.] Note that $H(a) = \sum_{j=1}^n |X_{j:n} - a|$ for all a. If $X_{k+1:n} > a > X_{k:n}$ for some k, then
>
> $$H(a) = \sum_{j=1}^k (a - X_{j:n}) + \sum_{j=k+1}^n (X_{j:n} - a),$$
>
> so that $H'(a) = 2k - n$ for all $a \in (X_{k+1:n}\,, X_{k:n})$. In other words, $H(a)$ is increasing for all $a \in (X_{k+1:n}, X_{k:n})$ if and only if $2k \geq n$; that is, $k \geq n/2$. In other words, H increases to the right of med and decreases to the left of med. This is another way to say that the function $H(a)$ is minimized at $a = \mathsf{med}$.

We find now that $\hat{\eta} = \mathsf{med}$.

 # 10. (b) Here, $f(x) = (2\theta)^{-1}\exp(-|x-\eta|/\theta)$. Therefore,

$$L(\theta\,, \eta) = (2\theta)^{-n}\exp\left(-\frac{1}{\theta}\sum_{j=1}^n |X_j - \eta|\right) \Rightarrow \log L(\theta\,, \eta) = -n\log 2 - n\log\theta - \frac{1}{\theta}\sum_{j=1}^n |X_j - \eta|.$$

Just as before, $\hat{\eta} = \mathsf{med}$; therefore,

$$\log L(\theta\,, \hat{\eta}) = -n\log 2 - n\log\theta - \frac{1}{\theta}\sum_{j=1}^n |X_j - \mathsf{med}|.$$

Note that

$$\frac{\partial}{\partial\theta}\log L(\theta\,, \hat{\eta}) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{j=1}^n |X_j - \mathsf{med}|.$$

Set this equal to zero to see that $\hat{\theta} = n^{-1} \sum_{j=1}^{n} |X_j - \mathsf{med}|$.

 # 12. Let $Y_j := e^{X_j}$, so that $Y_1, \ldots, Y_n$ is an independent sample from $\mathrm{LOGN}(\mu\,, \sigma^2)$.

(a) In terms of the X's, $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = n^{-1} \sum_{j=1}^{n} (X_j - \bar{X})^2$. Therefore, by the likelihood principle,

$$\hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} \ln Y_j, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{j=1}^{n} \left(\ln Y_j - \hat{\mu}\right)^2 .$$

(b) Since

$$E(Y) = E\left[e^X\right] = \exp\left(\mu + \frac{1}{2}\sigma^2\right),$$

the MLE for $E(Y)$ is

$$\widehat{E(Y)} = \exp\left(\hat{\mu} + \frac{1}{2}\hat{\sigma}^2\right).$$