Math 5080–1
Solutions to homework 4

June 19, 2012

 # 8. Notice that $P\{Y<1\}>0$ and $P\{X>1\}>0$; therefore, there is positive probability that $X<1$ *and* $Y>1$, by independence. If $X>1$ and $Y<1$, and this can happen with positive probability, then certainly $Y-X<0$; that is, $P\{Y-X<0\} \ge P\{Y<1\,,\ X>1\}>0$. It follows that $Y-X$ cannot have a χ^2 distribution, since χ^2-distributed random variables are >0 with probability one.

 # 10. We are interested in finding c that satisfies

$$0.95 = P\left\{\bar{X} < \frac{\theta}{c}\right\} = P\left\{\sum_{i=1}^{15} X_i < \frac{15\theta}{c}\right\} = P\left\{\frac{2}{\theta}\sum_{i=1}^{15} X_i < \frac{30}{c}\right\}.$$

Since $\sum_{i=1}^{15} X_i \sim \text{Gamma}(\theta\,,15)$, it follows that $(2/\theta)\sum_{i=1}^{15} X_i \sim \chi^2(30)$; therefore, we want to ensure that

$$0.95 = P\left\{\chi^2(30) < \frac{30}{c}\right\} \quad\Leftrightarrow\quad \frac{30}{c} \approx 43.77 \quad \text{(Table 4)} \quad\Leftrightarrow\quad c \approx \frac{30}{43.77} \approx 0.6854.$$

 # 12. First we standardize: $Z_1 := X_1/5$ and $Z_2 := X_2/5$ are independent $\mathrm{N}(0\,,1)$. Therefore, $D = 5\sqrt{Z_1^2+Z_2^2} \sim 5\times\sqrt{\chi^2(2)}$, whence

$$P\left\{D \le 12.25\right\} = P\left\{\sqrt{\chi^2(2)} \le \frac{12.25}{5}\right\} \approx P\left\{\chi^2(2) \le 6\right\}.$$

According to Table 5, $P\{\chi^2(2) < 5.8\} \approx 0.945$ and $P\{\chi^2(2) < 6.2\} \approx 0.955$; therefore, $P\{D \le 12.25\}$ is between 0.945 and 0.955 [approx.].

 # 14. We use the method of cdf's from 5010: For all $a \ge 0$,

$$F_{T^2}(a) = P\left\{T^2 \le a\right\} = P\left\{T \le \sqrt{a}\right\},$$

and $F_{T^2}(a) = 0$ since $T^2 \ge 0$. Differentiate $[da]$ to obtain the following:

$$f_{T^2}(a) = f_T\left(\sqrt{a}\right) \times \frac{d\sqrt{a}}{da} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1+\frac{a}{\nu}\right)^{-(\nu+1)/2} \cdot I\{a>0\} \times \frac{1}{2\sqrt{a}}.$$

In order to understand what this is, observe that

$$f_{T^2}(a) = \text{const} \cdot a^{-1/2} \left(1 + \frac{a}{\nu}\right)^{-(\nu+1)/2} \times I\{a > 0\},$$

where the constant is chosen to make the total area one. The $F(\nu_1, \nu_2)$ PDF is

$$f(a) = \text{const} \cdot a^{(\nu_1/2)-1} \left(1 + \frac{\nu_1}{\nu_2} a\right)^{-(\nu_1+\nu_2)/2} \times I\{a > 0\}.$$

If we set $\nu_1 = 1$ and $\nu_2 = \nu$, then the preceding 2 displays match. That is, $[t(\nu)]^2 = F(1, \nu)$.

 # 18. (a) $V_1 + V_2 \sim \chi^2(14)$; therefore, Table 5 shows that

$$P\{V_1 + V_2 \le 8.6\} = P\left\{\chi^2(14) \le 8.6\right\} \approx 0.144.$$

(b) Because $Z/\sqrt{V_1/5} \sim t(5)$, Table 6 tells us that

$$P\left\{\frac{Z}{\sqrt{V_1/5}} < 2.015\right\} = P\{t(5) < 2.015\} \approx 0.95.$$

(c) We know that $Z/\sqrt{V_2/9} \sim t(9)$; therefore, in accord with Table 6,

$$P\left\{Z > 0.611\sqrt{V_2}\right\} = P\left\{Z > 1.833\sqrt{\frac{V_2}{9}}\right\} = P\{t(9) > 1.833\} \approx 1 - 0.95 = 0.05.$$

(d) We know that $(V_1/5)/(V_2/9) \sim F(5, 9)$. Therefore, we write

$$P\left\{\frac{V_1}{V_2} < 1.450\right\} = P\left\{\frac{V_1/5}{V_2/9} < \frac{1.450 \times 9}{5}\right\} = P\{F(5, 9) < 2.61\} \approx 0.91,$$

thanks to Table 7.

(e) Since V_1 and V_2 are positive random variables, $0 < V_1/(V_1 + V_2) < 1$; therefore, the number b in question is necessarily between zero and one. Now, $V_1/(V_1 + V_2) < b$ means the same thing as $1 + (V_2/V_1) > (1/b)$; therefore,

$$0.90 = P\left\{\frac{V_2}{V_1} > \frac{1-b}{b}\right\} = P\left\{\frac{V_2/9}{V_1/5} > \frac{5(1-b)}{9b}\right\} = P\left\{F(9, 5) > \frac{5-5b}{9b}\right\}.$$

We would like to use Table 7, and cannot as is. Therefore, we observe that $F(9, 5) = 1/F(5, 9)$, and hence

$$0.90 = P\left\{F(5, 9) < \frac{9b}{5-5b}\right\},$$

which, according to Table 7, means that

$$\frac{9b}{5-5b} \approx 3.32 \iff b \approx \frac{16.6}{25.6} \approx 0.6484.$$

 # 24. We compute directly the MGF:

$$
\begin{aligned}
M_{(Y_n-n)/\sqrt{2n}}(t) = E\exp\left(\frac{t(Y_n-n)}{\sqrt{2n}}\right) &= \exp\left(-\frac{tn}{\sqrt{2n}}\right) E\exp\left(\frac{tY_n}{\sqrt{2n}}\right) \\
&= \exp\left(-t\sqrt{\frac{n}{2}}\right) \cdot M_{Y_n}\left(t/\sqrt{2n}\right) \\
&= \exp\left(-t\sqrt{\frac{n}{2}}\right) \cdot \left(1 - 2\frac{t}{\sqrt{2n}}\right)^{-n/2} \\
&= \exp\left(-t\sqrt{\frac{n}{2}}\right) \cdot \left(1 - t\sqrt{\frac{2}{n}}\right)^{-n/2},
\end{aligned}
$$

provided that $-\infty < t < \sqrt{n/2}$, and the preceding is $= \infty$ for all other values of t. Therefore,

$$
\begin{aligned}
\ln M_{(Y_n-n)/\sqrt{2n}}(t) &= -t\sqrt{\frac{n}{2}} - \frac{n}{2}\ln\left(1 - t\sqrt{\frac{2}{n}}\right) \\
&= -t\sqrt{\frac{n}{2}} - \frac{n}{2}\left(-t\sqrt{\frac{2}{n}} + \frac{t^2}{n}\right),
\end{aligned}
$$

since $\ln(1-z) = -z + (z^2/2) + \cdots$ when z is small [Taylor expansion]. This shows that $\ln M_{(Y_n-n)/\sqrt{2n}}(t) \to -t^2/2$ as $n \to \infty$, and the convergence theorem for MGFs does the rest.