Math 5080–1
Solutions to homework 2

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# 7, p. 166. We need \( \sum x_1 \sum x_2 f(x_1, x_2) \) to be one. That sum is equal to

\[
1 = f(0, 0) + f(0, 1) + f(0, 2) + f(1, 0) + f(1, 1) + f(1, 2) + f(2, 0) + f(2, 1) + f(2, 2)
\]

\[
= 0 + c + 2c + c + 2c + 3c + 2c + 3c + 4c
\]

\[
= 18c \quad \Rightarrow \quad c = \frac{1}{18}.
\]

# 8, p. 166.

(a) We want \( \sum x \sum y f(x, y) = 1 \); that is,

\[
1 = e \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \frac{2^x y!}{2^x x!} = e \sum_{x=0}^{\infty} \frac{2^x}{x!} \sum_{y=0}^{\infty} \frac{2^y}{y!} = ce^4 \Rightarrow c = e^{-4}.
\]

(b) The marginal PMF of \( X \) is: If \( x \) is not a positive integer then \( f_X(x) = 0 \); else

\[
f_X(x) = \sum_{y=0}^{\infty} f(x, y) = e^{-x} \sum_{y=0}^{\infty} \frac{2^y}{y!} = e^{-4} \frac{2^x}{x!} e^2 = e^{2x} / x!.
\]

That is, \( X \sim \text{Poisson}(2) \). Similarly, \( Y \sim \text{Poisson}(2) \).

(c) Yes. They are independent since the marginal PMF factorizes as a function of \( x \) only times a function of \( y \) only.

# 14, p. 167.

(a) \( f_1(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{0}^{\infty} e^{-(x+y)} \, dy \cdot I\{x > 0\} = e^{-x} I\{x > 0\} \).

Similarly, \( f_2(y) = e^{-y} I\{y > 0\} \). This shows that \( X \sim \text{Exp}(1), Y \sim \text{Exp}(1) \), and \( X \) and \( Y \) are independent [by factorization].

(b) \( F(x, y) = P\{X \leq x, Y \leq y\} \). This probability is zero unless both \( x \) and \( y \) are \( \geq 0 \). In the case that \( x \geq 0 \) and \( y \geq 0 \),

\[
F(x, y) = \int_{a \leq x} \int_{b \leq y} f(a, b) \, da \, db = \int_{0}^{x} e^{-a} \, da \int_{0}^{y} e^{-b} \, dy = (1 - e^{-x}) (1 - e^{-y}).
\]
(c) \[ P\{X > 2\} = \int_2^\infty f_1(x) \, dx = \int_2^\infty e^{-x} \, dx = e^{-2}. \]

(d) \[ P\{X < Y\} = \int_{0 < x < y} e^{-x}e^{-y} \, dx \, dy = \int_0^\infty e^{-x} (\int_x^\infty e^{-y} \, dy) \, dx = \int_0^\infty e^{-2x} \, dx = \frac{1}{2}. \]

(e) Perhaps it is easier to first compute \( P\{X + Y \leq 2\} \), since it is
\[
P\{X + Y \leq 2\} = \int_0^2 \left( \int_0^{2-x} e^{-y} \, dy \right) e^{-x} \, dx = \int_0^2 (1 - e^{-2}) e^{-x} \, dx = \int_0^2 e^{-x} \, dx - \int_0^2 e^{-2} \, dx,
\]
which is \( 1 - 3e^{-2} \). Therefore, \( P\{X + Y > 2\} = 3e^{-2} \).

(f) Yes, by factorization.

# 16, p. 167. The roots of this quadratic equation are
\[
t = \frac{-2X \pm \sqrt{4X^2 - 4Y}}{2} = -X \pm \sqrt{X^2 - Y}.
\]
The roots are real if and only if \( X^2 \geq Y \), which has probability
\[
P\{X^2 \geq Y\} = \int_{-1}^1 \left( \int_0^{x^2} f_Y(y) \, dy \right) f_X(x) \, dx = \int_{-1}^1 \frac{x^2}{2} \, dx = \frac{1}{3}.
\]

# 19, p. 168.

(a) We want
\[
1 = k \int_0^1 \left( \int_x^1 (x + y) \, dy \right) \, dx = k \int_0^1 \left[ x(1 - x) + \frac{1}{2}(1 - x^2) \right] \, dx.
\]
The integral is equal to
\[
\int_0^1 \left[ x - \frac{3x^2}{2} + \frac{1}{2} \right] \, dx = \frac{1}{2}.
\]
Therefore, \( k = 2 \).

(b) \( f_1(x) = \int_x^1 f(x, y) \, dy \cdot I\{0 \leq x \leq 1\} = 2 \int_x^1 (x+y) \, dy \cdot I\{0 \leq x \leq 1\} \), which is
\[
f_1(x) = (2x + 1 - 3x^2) \cdot I\{0 \leq x \leq 1\}.
\]
Similarly, \( f_2(y) = 2 \int_0^y (x+y) \, dx \cdot I\{0 \leq y \leq 1\} = 3y^2 \cdot I\{0 \leq y \leq 1\} \).

(c) There are several cases; you need to draw the region of integration in each case in order to follow the computations correctly:

- If \( 0 \leq x \leq y \leq 1 \), then
\[
F(x, y) = \int_0^x \left( \int_b^y f(a, b) \, da \right) \, db = 2 \int_0^x \left( \int_0^y (a + b) \, da \right) \, db
= 2 \int_0^x \left( \frac{y^2 - b^2}{2} + b(y - b) \right) \, db = y^2x + yx^2 - x^3.
\]
\( F(x, y) = \int_0^y \left( \int_b^y f(a, b) \, da \right) \, db = 2 \int_0^y \left( \int_b^y (a + b) \, da \right) \, db \\
= 2 \int_0^y \left( \frac{y^2 - b^2}{2} + b(y - b) \right) \, db = y^3. 
\)

\( F(x, y) = \int_0^x \left( \int_a^1 f(a, b) \, db \right) \, da = 2 \int_0^x \left( \int_a^1 (a + b) \, db \right) \, da \\
= 2 \int_0^x \left( a(1-a) + \frac{1-a^2}{2} \right) \, da = x^2 + x - x^3. 
\)

\( F(x, y) = 1 \) if \( x, y \geq 1 \) [both], then

\( F(x, y) = 0 \) in all other cases.

(d) Note that if \( 0 \leq x \leq 1 \), then

\[ f_X(x) = 2 \int_x^1 (x+y) \, dy = 2x(1-x) + 1 - x^2 = 2x + 1 - 3x^2. \]

Therefore, by definition [as a function of \( y \)],

\[ f_{Y|X}(y|x) = \frac{2(x+y)}{2x+1-3x^2}I\{x \leq y \leq 1\}. \]

(e) Note that if \( 0 \leq y \leq 1 \), then

\[ f_Y(y) = 2 \int_0^y (x+y) \, dx = 3y^2. \]

Therefore, by definition [as a function of \( x \)],

\[ f_{X|Y}(x|y) = \frac{2(x+y)}{3y^2}I\{0 \leq x \leq y\}. \]