Math 5080–1
Solutions to homework 2

May 29, 2012

 # 7, p. 166. We need $\sum_{x_1}\sum_{x_2} f(x_1\,,x_2)$ to be one. That sum is equal to

$$\begin{aligned}
1 &= f(0\,,0)+f(0\,,1)+f(0\,,2)+f(1\,,0)+f(1\,,1)+f(1\,,2)+f(2\,,0)+f(2\,,1)+f(2\,,2)\\
&= 0+c+2c+c+2c+3c+2c+3c+4c\\
&= 18c \quad\Rightarrow\quad c={}^1\!/\!_{18}.
\end{aligned}$$

 # 8, p. 166.

(a) We want $\sum_x\sum_y f(x\,,y)=1$; that is,

$$1=c\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{2^{x+y}}{x!y!}=c\sum_{x=0}^{\infty}\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!}=ce^2e^2=ce^4 \quad\Rightarrow\quad c=e^{-4}.$$

(b) The marginal PMF of X is: If x is not a positive integer then $f_X(x)=0$; else

$$f_X(x)=\sum_{y=0}^{\infty}f(x\,,y)=e^{-4}\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!}=e^{-4}\frac{2^x}{x!}e^2=\frac{e^2 2^x}{x!}.$$

That is, $X\sim\text{Poisson}(2)$. Similarly, $Y\sim\text{Poisson}(2)$.

(c) Yes. They are independent since the marginal PMF factorizes as a function of x only times a function of y only.

 # 14, p. 167.

(a) $f_1(x)=\int_{-\infty}^{\infty}f(x\,,y)\,dy=\int_0^{\infty}e^{-(x+y)}\,dy\cdot I\{x>0\}=e^{-x}I\{x>0\}$. Similarly, $f_2(y)=e^{-y}I\{y>0\}$. This shows that $X\sim\text{Exp}(1)$, $Y\sim\text{Exp}(1)$, and X and Y are independent [by factorization].

(b) $F(x\,,y)=P\{X\le x\,,Y\le y\}$. This probability is zero unless both x and y are ≥ 0. In the case that $x\ge 0$ and $y\ge 0$,

$$F(x\,,y)=\int\int_{\substack{a\le x\\ b\le y}}f(a\,,b)\,da\,db=\int_0^x e^{-a}\,da\int_0^y e^{-b}\,dy=\left(1-e^{-x}\right)\left(1-e^{-y}\right).$$

(c) $P\{X > 2\} = \int_2^\infty f_1(x)\,dx = \int_2^\infty e^{-x}\,dx = e^{-2}$.

(d) $P\{X < Y\} = \iint_{0<x<y} e^{-x}e^{-y}\,dx\,dy = \int_0^\infty e^{-x}(\int_x^\infty e^{-y}\,dy)\,dx = \int_0^\infty e^{-2x}\,dx = {}^1\!/_2$.

(e) Perhaps it is easier to first compute $P\{X+Y \le 2\}$, since it is

$$P\{X+Y \le 2\} = \int_0^2 \left(\int_0^{2-x} e^{-y}\,dy\right) e^{-x}\,dx = \int_0^2 \left(1 - e^{x-2}\right) e^{-x}\,dx = \int_0^2 e^{-x}\,dx - \int_0^2 e^{-2}\,dx,$$

which is $1 - 3e^{-2}$. Therefore, $P\{X+Y > 2\} = 3e^{-2}$.

(f) Yes, by factorization.

 # 16, p. 167. The roots of this quadratic equation are

$$t = \frac{-2X \pm \sqrt{4X^2 - 4Y}}{2} = -X \pm \sqrt{X^2 - Y}.$$

The roots are real if and only if $X^2 \ge Y$, which has probability

$$P\{X^2 \ge Y\} = \int_{-1}^1 \left(\int_0^{x^2} f_Y(y)\,dy\right) f_X(x)\,dx = \int_{-1}^1 \frac{x^2}{2}\,dx = \frac{1}{3}.$$

 # 19, p. 168.

(a) We want

$$1 = k\int_0^1 \left(\int_x^1 (x+y)\,dy\right) dx = k\int_0^1 \left[x(1-x) + \frac{1}{2}(1-x^2)\right] dx.$$

The integral is equal to

$$\int_0^1 \left[x - \frac{3x^2}{2} + \frac{1}{2}\right] dx = \frac{1}{2}.$$

Therefore, $k = 2$.

(b) $f_1(x) = \int_x^1 f(x\,,y)\,dy \cdot I\{0 \le x \le 1\} = 2\int_x^1 (x+y)\,dy \cdot I\{0 \le x \le 1\}$, which is

$$f_1(x) = \left(2x + 1 - 3x^2\right) \cdot I\{0 \le x \le 1\}.$$

Similarly, $f_2(y) = 2\int_0^y (x+y)\,dx \cdot I\{0 \le y \le 1\} = 3y^2 \cdot I\{0 \le y \le 1\}$.

(c) There are several cases; you need to draw the region of integration in each case in order to follow the computations correctly:

- If $0 \le x \le y \le 1$, then

$$\begin{aligned} F(x\,,y) &= \int_0^x \left(\int_b^y f(a\,,b)\,da\right) db = 2\int_0^x \left(\int_b^y (a+b)\,da\right) db \\ &= 2\int_0^x \left(\frac{y^2 - b^2}{2} + b(y-b)\right) db = y^2x + yx^2 - x^3. \end{aligned}$$

- If $0 \le y \le x$, then

$$F(x\,,y) = \int_0^y \left(\int_b^y f(a\,,b)\,da \right) db = 2\int_0^y \left(\int_b^y (a+b)\,da \right) db$$
$$= 2\int_0^y \left(\frac{y^2-b^2}{2} + b(y-b) \right) db = y^3.$$

- If $0 \le x \le 1$ and $y \ge 1$, then

$$F(x\,,y) = \int_0^x \left(\int_a^1 f(a\,,b)\,db \right) da = 2\int_0^x \left(\int_a^1 (a+b)\,db \right) da$$
$$= 2\int_0^x \left(a(1-a) + \frac{1-a^2}{2} \right) da = x^2 + x - x^3.$$

- If $x, y \ge 1$ [both], then $F(x\,,y) = 1$.
- In all other cases, $F(x\,,y) = 0$.

(d) Note that if $0 \le x \le 1$, then

$$f_X(x) = 2\int_x^1 (x+y)\,dy = 2x(1-x) + 1 - x^2 = 2x + 1 - 3x^2.$$

Therefore, by definition [as a function of y],

$$f_{Y|X}(y\,|\,x) = \frac{2(x+y)}{2x+1-3x^2} I\{x \le y \le 1\}.$$

(e) Note that if $0 \le y \le 1$, then

$$f_Y(y) = 2\int_0^y (x+y)\,dx = 3y^2.$$

Therefore, by definition [as a function of x],

$$f_{X|Y}(x\,|\,y) = \frac{2(x+y)}{3y^2} I\{0 \le x \le y\}.$$