Math 5050–1, Spring 2009
Homework 6 (Final)

Announcements. There will be no lecture on Monday April 27th, and Wednesday April 29th. This assignment is due on Wed. April 29th, and is **to be turned in to the Math Department** (JWB, ground level) before the end of Wednesday April 29th. I will have the assignments mailed to me in France the following Thursday, early in the morning. Therefore, late assignments cannot possibly be accepted. Please do not put the assignments in my mailbox, or under my office door; if you do, then you can expect your grade to come in after I return, which will be late in May.

1. (Theory question) Let $\{X_t\}_{t\geq 0}$ denote 1-dimensional Brownian motion throughout. You will need the following form of Itô's formula, discussed in the lectures:
$$f(t\,,X_t) = f(0\,,X_0) + M_t + \int_0^t \left(\dot{f}(s\,,X_s) + \frac{1}{2}f''(s\,,X_s)\right)\,ds,$$
where $\{M_t\}_{t\geq 0}$ is a continuous, mean-zero martingale.

   (a) Let $\lambda > 0$ be fixed. Use Itô's formula to show that
   $$f(t\,,X_t) = \exp\left(\lambda X_t - \frac{\lambda^2 t}{2}\right)$$
   defines a martingale, where $f(t\,,x) := \exp(\lambda x - \frac{1}{2}\lambda^2 t)$.

   (b) Let $X_0 := 0$, and $T := \min\{t > 0 : |X_t| = 1\}$ denote the first time Brownian motion leaves the interval $[-1\,,1]$. Show that
   $$E\left(e^{\lambda X_T - \frac{1}{2}\lambda^2 T}\right) = 1.$$

   (c) Show also that
   $$E\left(e^{-\lambda X_T - \frac{1}{2}\lambda^2 T}\right) = 1.$$
   (Hint: Redo (b), but start with $f(t\,,x) := \exp(-\lambda x - \frac{1}{2}\lambda^2 t)$.)

   (d) Conclude that
   $$E\left(\cosh\left(\lambda X_T\right) e^{-\frac{1}{2}\lambda^2 T}\right) = 1.$$
   (Hint: $2\cosh\theta := e^{\theta} + e^{-\theta}$.)

   (e) Deduce "Lévy's formula":
   $$E\left(e^{-\frac{1}{2}\lambda^2 T}\right) = \frac{1}{\cosh\lambda} \qquad \text{for all } \lambda > 0.$$
   Equivalently $[\theta := \frac{1}{2}\lambda^2]$,
   $$E\left(e^{-\theta T}\right) = \frac{1}{\cosh\left(\sqrt{2\theta}\right)} \qquad \text{for all } \theta > 0.$$

(f) We can compute the moments of T from the preceding; for instance, the first two moments are obtained by setting $\theta := 0$ in the following:

$$\frac{d}{d\theta} E\left(e^{-\theta T}\right) = -E\left(T e^{-\theta T}\right) \quad \text{and} \quad \frac{d^2}{d\theta^2} E\left(e^{-\theta T}\right) = E\left(T^2 e^{-\theta T}\right).$$

Use this and your answer to (e) to find $E(T)$ and $\text{Var}(T)$. [You know already that $E(T) = E(X_T^2) = 1$, using a different martingale computation.]

2. (Simulation question) Let $\{X_t\}_{t\geq 0}$ denote 1-dimensional Brownian motion. Consider the two curves defined by

$$f(t) := 1 + 2t^{1/3} \quad \text{and} \quad g(t) := -1 + t^{2/3}.$$

Let T denote the first time that Brownian motion hits either one of the two [it *will* hit them eventually, so this T is defined fine]. Simulate the probability that Brownian motion hits f before g. More precisely:

What is $P\{X_T = f(T)\}$?