Math 5040-1: Homework #3
The Markov-Chain Monte Carlo Method

1. Let $n \geqslant 1$ be a fixed integer. Let X be distributed uniformly at random on $\{1, \ldots, n\}$. This means: $P\{X = k\} = 1/n$ for $k = 1, \ldots, n$; and $P\{X = k\} = 0$ for all other values of k. Now let Y have the following conditional distribution: For each $k = 1, \ldots, n$ fixed,

$$P\{Y = j \,|\, X = k\} = \frac{1}{n-1},$$

for all $j = 1, \ldots, n$ with $j \neq k$; and $P\{Y = j \,|\, X = k\} = 0$ for all other values of j.

(a) Compute the joint mass function of (X, Y).

(b) Use **(a)** to describe an algorithm that selects a pair of distinct numbers from $\{1, \ldots, n\}$ such that all distinct pairs are equally likely to be chosen.

2. Let a be a permutation of $\{1, \ldots, n\}$; we can think of a as the n-tuple $(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are distinct elements in $\{1, \ldots, n\}$. Let b denote another permutation of $\{1, \ldots, n\}$. We say that a and b are *neighbors* if there exist two distinct integers i and j in $\{1, \ldots, n\}$ such that $a_k = b_k$ for all k in $\{1, \ldots, n\}$ except $k = i$ and $k = j$. For those two indices, $a_i = b_j$.

(a) Prove that, in the preceding definition, $a_j = b_i$ as well.

(b) How many neighbors does a given permutation a of $\{1, \ldots, n\}$ have?

(c) Let $\{X_n\}_{n=0}^{\infty}$ denote the simple walk on the permutations of $\{1, \ldots, n\}$. As always, this means that X always moves to any one of its neighbors with equal probability.

(i) Is $\{X_n\}_{n=0}^{\infty}$ irreducible? [Justify your assertions.]

(ii) Is it aperiodic? [Justify your assertions.]

(iii) Does $\{X_n\}_{n=0}^{\infty}$ have an invariant probability distribution? Find it, and/or justify your answer(s).

3. Find a Markov chain on the space of permutations of $\{1, \ldots, n\}$ such that for large k, $P\{X_k = a\} \approx 1/(n!)$ for all permutations a of $\{1, \ldots, n\}$.

4. Simulate this Markov chain 100 times for $n = 5$ and $k = 10$. Plot a histogram of the 100 different simulations of the X_{10}'s obtained in this way. Do you see a pattern? What does this all mean?