Final Exam Practice
Math 5010-1; July 31, 2015

1. Suppose X is a random variable with moment generating function,

$$M(t) = \frac{2}{3}e^{6t} + \frac{1}{6}e^{12t} + \frac{1}{6}e^{30t} \qquad \text{for all } t.$$

   (a) Is X continuous or discrete? Justify our assertion.

   (b) If X is continuous, then compute its density function. If X is discrete, then compute its mass function.

2. Suppose X takes the values 1, 2, and 5 with respective probabilities $2/3$, $1/6$, and $1/6$. Compute the moment generating function of $6X$.

3. Two fair dice are rolled independently. Calculate $\text{Var}(X_1X_2)$ where X_i denotes the number of dots rolled on the ith roll for $i = 1, 2$.

4. Let $\mathcal{R}$ be the collection of all points $(x\,,y)$ in the plane such that:

$$x \ge 0, \quad y \ge 0, \quad \text{and} \quad 4 \le x^2 + y^2 \le 9.$$

   Compute $\text{P}\{Y > 2X\}$ when $(X\,,Y)$ is drawn uniformly at random from the shaded region $\mathcal{R}$.

5. Let $\mathcal{R}$ be the collection of all points $(x\,,y)$ in the plane such that:

$$x \ge 0, \quad y \ge 0, \quad \text{and} \quad 4 \le x^2 + y^2 \le 9.$$

   Are X and Y independent? Prove or disprove.

6. Let X be a random variable with cumulative distribution function,

$$F(x) = \begin{cases} 1 & \text{if } x \ge 1, \\ x^2 & \text{if } 0 \le x < 1, \\ 0 & \text{if } x < 0. \end{cases}$$

   (a) Is X continuous or discrete? Justify your answer.

   (b) If your answer to (a) was "continuous," then compute the density function of X. Otherwise, compute the mass function of X.

   (c) What are:

i. $\mathrm{P}\{X \le 0.5\}$;
ii. $\mathrm{P}\{0.2 < X \le 1.5\}$;
iii. $\mathrm{P}\{0.2 \le X < 1.5\}$?

7. Let X be a random variable with cumulative distribution function,

$$F(x) = \begin{cases} 1 & \text{if } x \ge 1, \\ 1/2 & \text{if } 0 \le x < 1, \\ 0 & \text{if } x < 0. \end{cases}$$

(a) Is X continuous or discrete? Justify your answer.

(b) If you answered (a) as "continuous," then compute the density function of X. Else, compute the mass function of X.

(c) What are:

i. $\mathrm{P}\{X \le 0.5\}$;
ii. $\mathrm{P}\{0.2 < X \le 1\}$;
iii. $\mathrm{P}\{0.2 \le X < 1\}$?

8. Suppose that the random point $(X\,,Y)$ has joint density,

$$f(x\,,y) = \begin{cases} \frac{3}{2}\max(x\,,y) & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? Prove or disprove.

(b) Compute $\mathrm{P}\{X > 1/2\}$.

9. Let X be a random variable with density function,

$$f(x) = \begin{cases} \dfrac{\mathrm{e}^x}{\mathrm{e}-1} & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. Show the details of your computation.

(b) Compute the moment generating function of X.

10. Consider a random variable X that satisfies

$$\mathrm{P}\{X = x\} = \begin{cases} C2^{-x} & \text{if } x = 2, 3, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

where C is a finite and positive constant.

(a) Compute C. You may use, without proof the following fact from your calculus course: $1 + r + \cdots + r^n = (r^{n+1} - 1)/(r-1)$ for all $r > 0$ and integers $n \ge 0$.

(b) Calculate the cumulative distribution function of X.

(c) Prove that $\mathrm{P}\{140 < X \leq 213\} = 2^{-a} - 2^{-b}$ where a and b are positive integers, and compute a and b.

11. A certain basketball player makes n shot attempts. Her past history shows that she makes her shots 75% of the time, and misses 25% of the time. Assume that she shoots her n shots independently.

    (a) Suppose $n = 4$. Calculate the probability that she makes at least 90% of her shots, if $n = 4$.

    (b) Suppose $n = 400$. Use the central limit theorem to approximate the probability that she makes at least 90% of her shots.

12. Suppose X and Y are two independent random variables, both following an exponential distribution with parameter $\lambda = 1$.

    (a) What is the joint density function of (X, Y)?

    (b) What is the density function of $Z = \max(X, Y)$?

    (c) What is the expected value of $Z := \max(X, Y)$?

    (d) What is the variance of $Z := \max(X, Y)$?

13. Compute $\mathrm{E}(XY)$, where the random point (X, Y) has joint density,

$$f(x, y) = \begin{cases} \frac{3}{2}\max(x, y) & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

14. Let X and Y denote two independent random variables, each following a standard normal distribution. Define two new random variables,

$$U := \frac{X+Y}{\sqrt{2}}, \quad V := \frac{X-Y}{\sqrt{2}}.$$

    (a) Compute the joint density of U and V.

    (b) Compute the marginal densities of U and V.

    (c) Are U and V independent? Prove or disprove.

15. Let X and Y denote two independent random variables, each following a uniform distribution on $(0, 1)$. Define two new random variables,

$$U := \frac{X+Y}{\sqrt{2}}, \quad V := \frac{X-Y}{\sqrt{2}}.$$

    Are U and V independent? Prove or disprove.

16. Suppose we pick a random point (X, Y) uniformly on the circle of radius one about the origin $(0, 0)$ of the plane. Define two new random variables,

$$U := \frac{X+Y}{\sqrt{2}}, \quad V := \frac{X-Y}{\sqrt{2}}.$$

    Compute the joint density of (U, V).

17. Suppose X and Y are independent with respective densities f_X and f_Y.

    (a) Prove that the density of $Z := X + Y$ is

    $$f_Z(a) = \int_{-\infty}^{\infty} f_X(x) f_Y(a - x)\,\mathrm{d}x \qquad \text{for all } a.$$

    (b) Suppose X and Y are independent and have the common density

    $$f_X(b) = f_Y(b) = \begin{cases} \mathrm{e}^{-b} & \text{if } b > 0, \\ 0 & \text{otherwise.} \end{cases}$$

    Then use the previous part to compute the density of $Z := X + Y$.

18. Two fair dice are rolled independently. Let D_i denote the number of dots rolled by die number i, where $i = 1, 2$, and then set $X_1 := \max(D_1\,, D_2)$ and $X_2 := D_1 + D_2$.

    (a) Find the joint probability mass function of $(X_1\,, X_2)$.

    (b) Find $\mathrm{P}\{X_2 < 1.5X_1\}$.

19. Let X and Y be independent random variables with respective mass functions f_X and f_Y.

    (a) Prove that the mass function of $Z := X + Y$ is

    $$f_Z(a) = \sum_x f_X(x) f_Y(a - x) \qquad \text{for all } a.$$

    (b) Use part (a) to prove that if X has a Poisson distribution with parameter λ_1 and Y has a Poisson distribution with parameter λ_2, and if X and Y are independent, then $X + Y$ has a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

20. Prove that if X and Y are two independent random variables, both distributed uniformly on $(0\,, 1)$, then $X + Y$ is not uniformly distributed. [Hint. Examine the behavior of the function $M_{X+Y}(t) := \mathrm{E}[\mathrm{e}^{t(X+Y)}]$ as $t \to \infty$.]

TABLE 5.1 AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998