1. The geometric distribution, continued

1.1. An example. A couple has children until their first son is born. Suppose the sexes of their children are independent from one another [unrealistic], and the probability of girl is 0.6 every time [not too bad]. Let $X$ denote the number of their children to find then that $X = \text{Geom}(0.4)$. In particular,

$$P(X \leq 3) = f(1) + f(2) + f(3)$$
$$= p + p(1 - p) + p(1 - p)^2$$
$$= p [1 + 1 - p + (1 - p)^2]$$
$$= p [3 - 3p + p^2]$$
$$= 0.784.$$

1.2. The tail of the distribution. Now you may be wondering why these random variables are called “geometric.” In order to answer this, consider the tail of the distribution of $X$ (probability of large values). Namely, for all $n \geq 1$,

$$P(X \geq n) = \sum_{j=n}^{\infty} p(1 - p)^{j-1}$$
$$= p \sum_{k=n-1}^{\infty} (1 - p)^k.$$

Let us recall an elementary fact from calculus.
Lemma 9.1 (Geometric series). If $r \in (0, 1)$, then for all $n \geq 0$,
\[ \sum_{j=n}^{\infty} r^j = \frac{r^n}{1-r}. \]

Proof. Let $s_n = r^n + r^{n+1} + \cdots = \sum_{j=n}^{\infty} r^j$. Then, we have two relations between $s_n$ and $s_{n+1}$:

1. $r s_n = \sum_{j=n+1}^{\infty} r^j = s_{n+1}$; and
2. $s_{n+1} = s_n - r^n$.

Plug (2) into (1) to find that $r s_n = s_n - r^n$. Solve to obtain the lemma. □

Return to our geometric random variable $X$ to find that
\[ P(X \geq n) = p \frac{(1-p)^{n-1}}{1-(1-p)} = (1-p)^{n-1}. \]

That is, $P(X \geq n)$ vanishes geometrically fast as $n \to \infty$.

In the couples example (§1.1),
\[ P(X \geq n) = 0.6^{n-1} \quad \text{for all } n \geq 1. \]

2. The negative binomial distribution

Suppose we are tossing a $p$-coin, where $p \in (0, 1)$ is fixed, until we obtain $r$ heads. Let $X$ denote the number of tosses needed. Then, $X$ is a discrete random variable with possible values $r, r+1, r+2, \ldots$. When $r = 1$, then $X$ is Geom($p$). In general,
\[ f(x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & \text{if } x = r, r+1, r+2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \]

This $X$ is said to have a negative binomial distribution with parameters $r$ and $p$. Note that our definition differs slightly from that of your text (p. 117).

3. The Poisson distribution

Choose and fix a number $\lambda > 0$. A random variable $X$ is said to have the Poisson distribution with parameter $\lambda$ (written Poiss($\lambda$)) if its mass function is
\[ f(x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & \text{if } x = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \]
In order to make sure that this makes sense, it suffices to prove that
\[ \sum_x f(x) = 1, \]
but this is an immediate consequence of the Taylor expansion of \( e^x \), viz.,
\[ e^x = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}. \]

**3.1. Law of rare events.** Is there a physical manner in which \( \text{Poisson}(\lambda) \) arises naturally? The answer is “yes.” Let \( X = \text{Bin}(n, \lambda/n) \). For instance, \( X \) could denote the total number of sampled people who have a rare disease (population percentage = \( \lambda/n \)) in a large sample of size \( n \). Then, for all fixed integers \( k = 0, \ldots, n, \)
\[ f_X(k) = \binom{n}{k} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k}. \]

Poisson’s “law of rare events” states that if \( n \) is large, then the distribution of \( X \) is approximately \( \text{Poisson}(\lambda) \). In order to deduce this we need two computational lemmas.

**Lemma 9.2.** For all \( z \in \mathbb{R} \),
\[ \lim_{n \to \infty} \left( 1 + \frac{z}{n} \right)^n = e^z. \]

**Proof.** Because the natural logarithm is continuous on \((0, \infty)\), it suffices to prove that
\[ \lim_{n \to \infty} n \ln \left( 1 + \frac{z}{n} \right) = z. \]
By Taylor’s expansion,
\[ \ln \left( 1 + \frac{z}{n} \right) = \frac{z}{n} + \frac{\theta^2}{2}, \]
where \( \theta \) lies between 0 and \( z/n \). Equivalently,
\[ \frac{z}{n} \leq \ln \left( 1 + \frac{z}{n} \right) \leq \frac{z}{n} + \frac{z^2}{2n^2}. \]
Multiply all sides by \( n \) and take limits to find (9), and thence the lemma. \( \square \)

**Lemma 9.3.** If \( k \geq 0 \) is a fixed integer, then
\[ \binom{n}{k} \frac{n^k}{k!} \text{ as } n \to \infty. \]
where \( a_n \sim b_n \) means that \( \lim_{n \to \infty} (a_n/b_n) = 1. \)
Proof. If \( n \geq k \), then

\[
\frac{n!}{n^k(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{n^k} = \frac{n \times n-1 \times \cdots \times n-k+1}{n} \to 1 \quad \text{as} \quad n \to \infty.
\]

The lemma follows upon writing out \( \binom{n}{k} \) and applying the preceding to that expression. \( \square \)

Thanks to Lemmas 9.2 and 9.3, and to (8),

\[
f_X(k) \sim \frac{n^k \lambda^k}{k!} \frac{e^{-\lambda}}{n^k e^{-\lambda} \lambda^k} = \frac{e^{-\lambda} \lambda^k}{k!}.
\]

That is, when \( n \) is large, \( X \) behaves like a Poiss(\( \lambda \)), and this proves our assertion.