Solutions

Midterm #4
Mathematics 5010–1, Summer 2006
Department of Mathematics, University of Utah
July 17, 2006

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* This is a closed-book, closed-notes examination.
* This exam begins at 1:00 p.m. and ends at 2:00 p.m. sharp. There are 3 questions for a total of 40 points.
* Write your answers clearly. **If you show merely a numerical answer, then you are likely to receive zero partial credit.** So show and explain your work.
* **Confine your work to this worksheet.** You may use both sides of the paper. There is also an extra sheet of paper for you to write on if you wish.

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1. (25 points total) Suppose $(X\,,Y)$ have joint pdf

$$f(x\,,y) = \begin{cases} e^{-(x+y)} & \text{if } x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (15 points) Find the respective marginal pdfs of X and Y. Are X and Y independent? Explain your answer.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} e^{-x} & \text{if } x>0, \\ 0 & \text{o/w}. \end{cases}$$

f_Y is the same function by symmetry.

Because $f(x,y) = f_X(x)\,f_Y(y)$, $\boxed{X \text{ and } Y \text{ are independent}}$.

(b) (10 points) Find EX and $\mathrm{Var}X$.

$$EX = \int_0^\infty x e^{-x} dx = \Gamma(2) = 1.$$

EY is the same because

$$f_Y(a) = f_X(a) = \begin{cases} e^{-a} & a>0, \\ 0 & \text{o/w}. \end{cases}$$

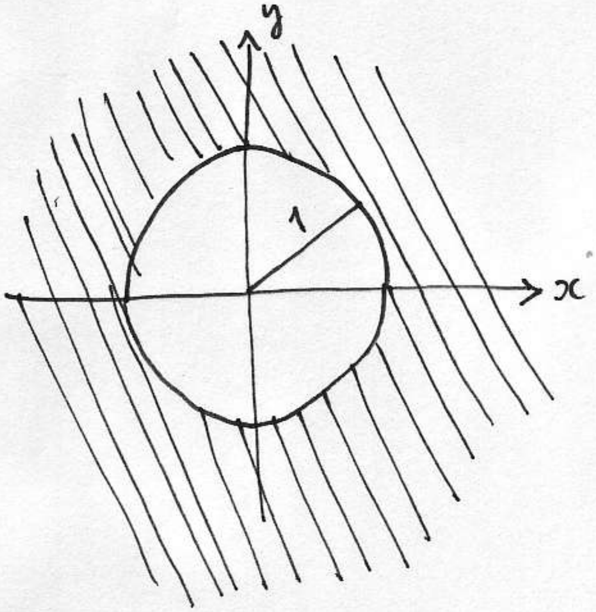
$$EX^2 = \int_0^\infty x^2 e^{-x} dx = \Gamma(3) = 2! = 2.$$

$$\Rightarrow \quad \mathrm{Var}X = \mathrm{Var}Y = 2 - 1^2 = 1.$$

2. (5 points) Suppose X and Y are independent standard normals. Recall that their common pdf is

$$f(a) = \frac{e^{-a^2/2}}{\sqrt{2\pi}} \qquad -\infty < a < \infty.$$

Find $P\{X^2 + Y^2 > 1\}$. (HINT: You do not need a normal table for this.)



$$P\{X^2+Y^2>1\} = P\left\{ (X,Y) \text{ falls in the shaded region} \right\}$$

$$= \iint_{x^2+y^2>1} \underbrace{\frac{e^{-(x^2+y^2)/2}}{2\pi}}_{f(x,y) = f_X(x) f_Y(y).} \, dx\, dy$$

Compute in polar coordinates:

$$" = \int_0^{2\pi} \int_1^{\infty} \frac{e^{-r^2/2}}{2\pi} \, r\, dr\, d\theta.$$

$$\int_1^{\infty} e^{-r^2/2} \, r\, dr = \int_{1/2}^{\infty} e^{-s}\, ds = e^{-1/2} = \frac{1}{\sqrt{e}} \qquad [s = r^2/2].$$

$$\Rightarrow \qquad P\{X^2+Y^2>1\} = \int_0^{2\pi} \frac{1}{\sqrt{e}} \cdot \frac{1}{2\pi}\, d\theta = \frac{1}{\sqrt{e}} \approx 0.606.$$

3. (10 points) Suppose X and Y are independent with distributions:

(a) $P\{X=0\}=\frac{1}{2}$ and $P\{X=1\}=\frac{1}{2}$.

(b) $P\{Y=1\}=\frac{1}{3}$ and $P\{Y=-1\}=\frac{2}{3}$.

Compute the mass function of $X+Y$.

Possible values of $X+Y$: $0, 1, -1, 2$

$$p_{X+Y}(1) = p(0,1) = p_X(0)\, p_Y(1) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}.$$

$$p_{X+Y}(2) = p(1,1) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$

$$p_{X+Y}(-1) = p(0,-1) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

$$p_{X+Y}(0) = p(1,-1) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}.$$