Solutions

Midterm #3

Mathematics 5010–1, Summer 2006

Department of Mathematics, University of Utah

July 5, 2006

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* This is a closed-book, closed-notes examination.
* This exam begins at 1:00 p.m. and ends at 2:00 p.m. sharp.
* This exam is made up of 4 questions for a total of 40 points.
* Write your answers clearly. **If you show merely a numerical answer, then you are likely to receive zero partial credit.** So show and explain your work.
* **Confine your work to this worksheet.** You may use both sides of the paper. There is also an extra sheet of paper for you to write on if you wish, and a **normal table at the very back.**

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1. (10 points total) Suppose that the height, in inches, of a 25-year-old man is a normal random variable with mean $\mu = 71$ and variance $\sigma^2 = 6.25$.

   (a) (5 points) What percentage of 25-year-old men are over 6 feet 2 inches tall?

$$P\{X \geqslant 74\} = 1 - \Phi\left(\frac{74-71}{\sqrt{6.25}}\right) \simeq 1 - \Phi(1.2) \simeq 1 - 0.8849 = \boxed{0.1151}$$

   (b) (5 points) What percentage of men in the 6-footer club are over 6-foot 5 inches?

$$P\{X \geqslant 77 \mid X \geqslant 72\} = \frac{P\{X \geqslant 77\}}{P\{X \geqslant 72\}} = \frac{1 - \Phi\left(\frac{77-71}{\sqrt{6.25}}\right)}{1 - \Phi\left(\frac{72-71}{\sqrt{6.25}}\right)}$$

$$= \frac{1-\Phi(2.4)}{1-\Phi(0.4)} \simeq \frac{1-0.9918}{1-0.6554} \simeq \boxed{0.02}$$

2. (10 points total) For all $a, b > 0$ define

$$B(a\,,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

(a) (5 points) Prove that if $a, b > 0$ are fixed parameters, then the following is a probability density function:

$$f(x) := \begin{cases} \dfrac{1}{B(a\,,b)} x^{a-1}(1-x)^{b-1} & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

- $f(x) \geq 0$
- $\int_0^1 f(x)\,dx = \dfrac{1}{B(a,b)} \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = 1.$

(b) (5 points) Compute $E[X]$.

$$E[X] = \int_0^1 x \cdot \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{1}{B(a,b)} \int_0^1 x^{(a+1)-1}(1-x)^{b-1}\,dx$$

$$= \frac{B(a+1,b)}{B(a,b)}.$$

3. (10 points) If $X$ is uniformly distributed on $(0, 1)$ then what is the density function of $Y := e^X$?

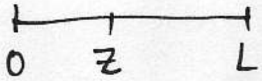
$$F_Y(a) = P\{e^X \le a\} = P\{X \le \ln a\} \qquad \text{if } a > 0$$

$$= \begin{cases} \ln a & 0 \le \ln a \le 1 \\ 1 & \ln a > 1 \\ 0 & \ln a < 0 \end{cases}$$

[why?]

$$\Rightarrow \quad f_Y(a) = \begin{cases} 1/a & 0 \le \ln a \le 1 \\ 0 & \text{o/w} \end{cases}$$

$$= \begin{cases} 1/a & 1 \le a \le e, \\ 0 & \text{o/w} \end{cases}.$$




Let $Z$ = place of the break, so that

$Z$ is unif-$[0,L]$.

4. (10 points) There is a stick of length $L$ which we may think of, mathematically, as the interval $[0\,,L]$. Choose a point at random along the stick (e.g., let the point be uniformly distributed in the interval $[0\,,L]$), and break the stick at that point. Let $X$ denote the ratio of the length of the longer piece to the shorter one. Find $P\{X \le L/4\}$.

If $Z \le \frac{L}{2}$ then $X = \frac{L-Z}{Z} \le \frac{L}{4} \leftrightarrow Z \geqslant \frac{4L}{4+L}$.

If $Z > \frac{L}{2}$ then $X = \frac{Z}{L-Z} \le \frac{L}{4} \longleftrightarrow Z \le \frac{L^2}{4+L}$.

Now, if $L \le 4$ then $P\{X \le \frac{L}{4}\} = 0$ because $X \geq 1$.

On the other hand, if $L > 4$ then $\frac{4L}{4+L} > 1$ and $\frac{L^2}{4+L} < 1$.

Therefore in all cases,

$$\boxed{P\{X \le \frac{L}{4}\} = 0}.$$