Solution to HW 8 - 5010-1, Summer '06

(p 319 #54) $u = g_1(x,y) = xy$, $v = g_2(x,y) = x/y$

$\Rightarrow$ $x = h_1(u,v) = \sqrt{uv}$, $y = h_2(u,v) = \sqrt{u/v}$.

$$J(x,y) = \det \begin{bmatrix} y & x \\ 1/y & -x/y^2 \end{bmatrix} = -2x/y = -2v.$$

$$\Rightarrow f_{U,V}(u,v) = f_{X,Y}(x,y) \,/\, |J(x,y)|$$

$$= \frac{1}{x^2y^2} \times \frac{1}{2v} \qquad \text{if } x,y \geq 1$$

$$= \begin{cases} \dfrac{1}{2vu^2} & \text{if } uv \geq 1 \text{ and } u/v \geq 1 \\ 0 & \text{o/w.} \end{cases}$$

(p408 #4) Recall : $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ and $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

[Both are derived via induction].

Now,
$$E(|X-Y|) = \sum_{i,j=1}^{m} |i-j| \; P(X=i, Y=j)$$

$$= \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} |i-j|,$$

$$= \frac{2}{m^2} \sum_{i=1}^{m} \sum_{j=i+1}^{m} (j-i) \qquad \text{[by symmetry].}$$

Now $\sum_{j=i+1}^{m} (j-i) = \sum_{k=1}^{m-i} k = \frac{(m-i)(m-i+1)}{2} = \frac{1}{2}\left[(m-i)^2 + (m-i)\right]$

(HW8, p2)

Therefore,

$$\sum_{i=1}^{m} \sum_{j=i+1}^{m} (j-i) = \frac{1}{2} \sum_{i=1}^{m} (m-i)^2 + \frac{1}{2} \sum_{i=1}^{m} (m-i)$$

$$= \frac{1}{2} \sum_{j=1}^{m-1} j^2 + \frac{1}{2} \sum_{j=1}^{m-1} j$$

$$= \frac{1}{2}\left[\frac{(m-1)(m)(2(m-1)+1)}{6} + \frac{(m-1)(m)}{2} \right]$$

$$= \frac{1}{2}\left[\frac{(m-1)m(2m+1)}{6} + \frac{m(m-1)}{2} \right]$$

$$= \frac{m(m-1)}{4}\left[\frac{2m+1}{3} + 1 \right] = \frac{m(m-1)}{4} \cdot \frac{2m+4}{3}$$

$$= \frac{m(m-1)(\cancel{2}m+2)}{6}$$

$$\Rightarrow \quad E(|X-Y|) = \frac{(m-1)(m+2)}{3m} \qquad \text{[N.B. the typo]}$$

(p412, #30)

$$E[(X-Y)^2] = E(X^2) - 2E(XY) + E(Y^2)$$

$$= E(X^2) - 2EX\,EY + E(Y^2) \qquad \text{by indep.}$$

$EX = EY = \mu$

$E(X^2) = \sigma^2 + \mu^2 = E(Y^2)$. So,

$$E[(X-Y)^2] = \mu^2 + \sigma^2 - 2\mu^2 + \mu^2 + \sigma^2 = 2\sigma^2.$$

HW 8, p3

p412, #33 (a) $E\left[(2+X)^2\right] = 4 + 4EX + E(X^2)$

$$= 4 + 4\times 1 + (5 + 1^2) = 14$$

(b) $Var(4+3X) = 9 \, var X = 45.$

p412, #40 • $E(X) = \int_0^\infty \int_0^\infty x \frac{1}{y} e^{-(y+\frac{x}{y})} \, dx \, dy$

$$= \int_0^\infty \frac{e^{-y}}{y} \underbrace{\left(\int_0^\infty x e^{-x/y} \, dx \right)}_{y} dy$$

$$= \int_0^\infty e^{-y} \, dy = 1.$$

• $E(Y) = \int_0^\infty \int_0^\infty y \frac{1}{y} e^{-(y+\frac{x}{y})} \, dx \, dy$

$$= \int_0^\infty e^{-y} \underbrace{\left(\int_0^\infty e^{-(x/y)} \, dx \right)}_{y} dy$$

$$= \int_0^\infty y e^{-y} \, dy = \Gamma(2) = 1.$$

• $E(XY) = \int_0^\infty \int_0^\infty xy \frac{1}{y} e^{-(y + x/y)} \, dx \, dy$

$$= \int_0^\infty e^{-y} \underbrace{\left(\int_0^\infty x e^{-x/y} \, dx \right)}_{y^2} dy = \Gamma(3) = 2.$$

therefore, $Cov(X, Y) = 2 - EX\,EY = 1.$

HW 8, p4

p417, #75

$$M_X(t) = e^{2[e^t - 1]} \Rightarrow X = \text{Poisson}(2)$$

$$M_Y(t) = \left(\frac{3}{4}e^t + \frac{1}{4}\right)^{10} \Rightarrow Y = \text{Bin}(10, 3/4).$$

(a) $P[X+Y=2] = p(0,2) + p(1,1) + p(2,0)$

$$= P(X=0)\,P(Y=2) + P(X=1)\,P(Y=1) + P(X=2)\,P(Y=0)$$

$$= \binom{10}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^{10} \cdot \frac{e^{-2}2^2}{2!} + \binom{10}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^9 \frac{e^{-2}2}{1!}$$

$$+ \binom{10}{2}\left(3/4\right)^2\left(1/4\right)^8 \frac{e^{-2}2^0}{0!}.$$

... which can be computed, I suppose, on a calculator.

(b) $P[XY=0] = P(X=0)\,P(Y=0) = \binom{10}{0}(3/4)^0(1/4)^{10} \cdot \frac{e^{-2}2^0}{0!}$

$$= e^{-2}/4^{10}.$$

(c) $E(XY) = EX\,EY$ (independence)

$$= np \cdot \lambda$$

$$= 10 \times \frac{3}{4} \times 2 = 120/4 = 30.$$

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p417, #77(b)

$$M_X(t) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{e^{tx}\,e^{-y}\,e^{-(x-y)^2/2}}{\sqrt{2\pi}}\,dy\,dx$$

$\boxed{\text{HW 8, p5}}$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y} \Bigg[\underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left[(x-y)^2 - 2tx \right]\right) dx}_{\text{call this } \mathcal{I}.} \Bigg] dy .$$

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left[x^2 - 2xy - 2tx \right]\right) dx \times e^{-y^2/2}$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left[x^2 - 2(t+y)x \right]\right) dx \cdot e^{-y^2/2}$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left[x - 2(t+y)x + (t+y)^2 \right]\right) dx \cdot e^{-y^2/2} \cdot e^{(t+y)^2/2}$$

$$= \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2} \underbrace{(x - (t+y))^2}_{u^2} \Big) \underbrace{dx}_{du} \cdot \exp\left(-\frac{1}{2}\left[y^2 - (t+y)^2 \right]\right)$$

$$= \sqrt{2\pi} \exp\left(-\frac{1}{2}\left[\cancel{y^2} - t^2 - 2ty - \cancel{y^2} \right]\right)$$

$$= \sqrt{2\pi} \exp\left(\frac{t^2 + 2ty}{2} \right).$$

Therefore,

$$M_X(t) = \int_0^\infty \exp\left(-y + \frac{t^2}{2} + ty\right) dy$$

$$= e^{t^2/2} \int_0^\infty e^{-y(1-t)} dy = \begin{cases} \dfrac{e^{t^2/2}}{1-t} & \text{if } t<1 \\ \infty & \text{if } t \geq 1 \end{cases}$$

HW 8, p6

$$M_Y(t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{e^{ty}\, e^{-y}\, e^{-(x-y)^2/2}}{\sqrt{2\pi}}\, dy\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-y(1-t)} \underbrace{\left(\int_{-\infty}^{\infty} e^{-(x-y)^2/2}\, dx \right)}_{\sqrt{2\pi}} dy$$

$$= \begin{cases} \dfrac{1}{1-t} & \text{if } t < 1 \\ \infty & \text{if } t \geq 1 \end{cases}$$