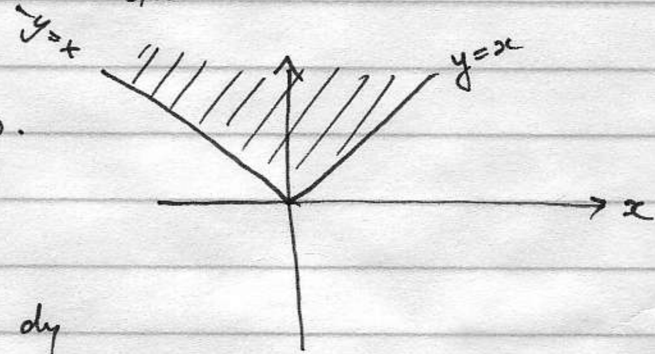
Solutions to Assignment 7

Math 5010, summer 2006

#8 p313

$$f(x,y) = \begin{cases} c(y^2-x^2)e^{-y} & -y \le x \le y, \quad 0<y \\ 0 & \text{o/w} \end{cases}$$



(a) $f(x,y) \geq 0$ just requires $c \geq 0$.

Check for $\iint f = 1$:

$$1 = \int_0^\infty \int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx\,dy$$

$$= c\int_0^\infty \underbrace{\left(\int_{-y}^{y} (y^2-x^2)e^{-y}\,dx\right)}_{I(y)} dy$$

$$I(y) = \int_{-y}^{y} y^2 e^{-y}\,dx - \int_{-y}^{y} x^2 e^{-y}\,dx$$

$$= 2y^3e^{-y} - \frac{2}{3}y^3e^{-y} = \frac{4}{3}y^3e^{-y}.$$

$$\Rightarrow 1 = c\int_0^\infty I(y)\,dy = \frac{4c}{3}\int_0^\infty y^3e^{-y}\,dy = \frac{4c}{3}\Gamma(4) = \frac{4c}{3}\times 3!$$
$$= 8c$$

$$\Rightarrow c = 1/8 \quad \text{and} \quad f(x,y) = \begin{cases} \frac{1}{8}(y^2-x^2)e^{-y} & \text{if } \begin{cases} -y \le x \le y \\ y>0 \end{cases} \\ 0 & \text{o/w} \end{cases}$$

(b)

$$f_Y(y) = \int_{-y}^{y} \frac{1}{8}(y^2-x^2)e^{-y}\,dx \qquad \text{all } y>0$$

$$\therefore\ f_Y(y) = \frac{1}{8} y^2 e^{-y} \int_{-y}^{y} dx - \frac{1}{8} e^{-y} \int_{-y}^{y} x^2\, dx$$

$$= \frac{1}{4} y^3 e^{-y} - \frac{1}{12} y^3 e^{-y} = \frac{1}{6} y^3 e^{-y}, \quad \text{if } y>0.$$

$$f_Y(y) = 0 \quad \text{if } y \le 0.$$

If $x>0$ then (see picture on page 1):

$$f_X(x) = \int_x^\infty \frac{1}{8}(y^2 - x^2) e^{-y}\, dy$$

$$= \frac{1}{8} \int_x^\infty y^2 e^{-y}\, dy - \frac{x^2}{8} \int_x^\infty e^{-y}\, dy.$$

$$\int_x^\infty y^2 e^{-y}\, dy = -y^2 e^{-y}\Big|_x^\infty + 2\int_x^\infty y e^{-y}\, dy$$

$$= x^2 e^{-x} + 2\int_x^\infty y e^{-y}\, dy.$$

$u = y^2$, $v' = e^{-y}$

$u' = 2y$, $v = -e^{-y}$

Integrate by parts again:

$$\int_x^\infty y e^{-y}\, dy = -y e^{-y}\Big|_x^\infty + \int_x^\infty e^{-y}\, dy$$

$u = y$ $\quad v' = e^{-y}$

$u' = 1$ $\quad v = -e^{-y}$

$$= x e^{-x} + e^{-x} = (1+x) e^{-x}.$$

$$\Rightarrow \int_x^\infty y^2 e^{-y}\, dy = (x^2 + 2x + 2) e^{-x}$$

$$\Rightarrow f_X(x) = \frac{1}{8}(x^2 + 2x + 2) e^{-x} - \frac{1}{8} x^2 e^{-x} = \frac{x+1}{4} e^{-x} \quad \text{if } x>0.$$

In general, $\quad f_X(x) = \dfrac{|x|+1}{4} e^{-|x|}.$

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p3 (c) $EX = 0$ by symmetry.

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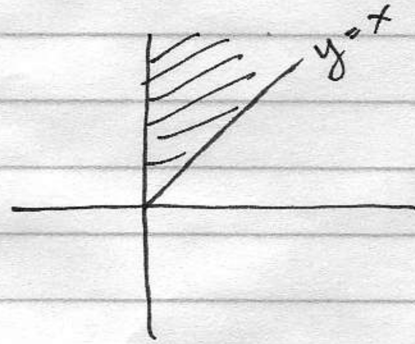
$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } x, y \geq 0 \\ 0 & \text{o/w} \end{cases}$$

(a) $P\{X < Y\} = P\{(X,Y) \text{ in the shaded region}\}$



$$= \int_0^\infty \int_0^y e^{-(x+y)}\, dx\, dy$$

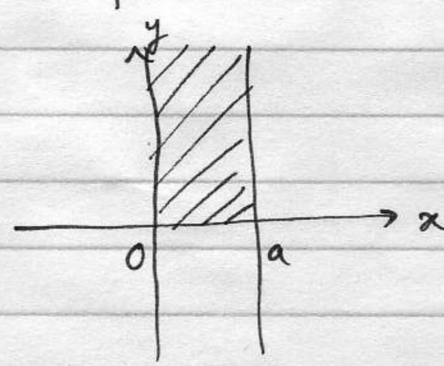
$$= \int_0^\infty e^{-y} \left(\int_0^y e^{-x} dx \right) dy$$

$$= \int_0^\infty e^{-y} (1 - e^{-y})\, dy$$

$$= \int_0^\infty e^{-y} dy - \int_0^\infty e^{-2y} dy = \frac{1}{2}.$$

(b) $P\{X < a\} = P\{(X,Y) \text{ in the shaded region}\}$

$$= \int_0^\infty \int_0^a e^{-(x+y)} dx\, dy \quad \text{if } a > 0$$



$$= \begin{cases} 1 - e^{-a} & \text{if } a > 0 \\ 0 & \text{o/w} \end{cases}$$

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ⓐ $F_{A,B,C}(a,b,c) = \begin{cases} abc & \text{if } 0 \le a \le 1,\ 0 \le b \le 1,\ 0 \le c \le 1, \\ 0 & \text{o/w.} \end{cases}$

ⓑ $P\{\text{roots real}\} = P\{B^2 \geq 4AC\} = P\{B^2 \geq 4AC,\quad AC \leq 1/4\}$

$$= \int_0^1 \int_0^{\min\left(\frac{1}{4c},1\right)} \left(\int_{2\sqrt{ac}}^1 1 \, d\cancel{b} \right) da \, dc$$

$$= \int_0^1 \int_0^{\min\left(\frac{1}{4c},1\right)} \left(1 - 2\sqrt{ac}\right) da \, dc.$$

$$= \int_{1/4}^1 \int_0^{\frac{1}{4c}} \left(1 - 2\sqrt{ac}\right) da \, dc + \int_0^{1/4} \int_0^1 \left(1 - 2\sqrt{ac}\right) da \, dc.$$

$$\int (1 - 2\sqrt{ac}) \, da = 1 - 2\sqrt{c} \int a^{\frac{1}{2}} \, da$$

$$= 1 - \frac{4\sqrt{c}}{3} a^{3/2}.$$

$$\Rightarrow \quad \int_0^{\frac{1}{4c}} (1 - 2\sqrt{ac}) da = \frac{1}{4c} - \frac{4\sqrt{c}}{3} \left(\frac{1}{4c}\right)^{3/2}$$

$$= \frac{1}{4c} - \frac{1}{6c} = \frac{1}{12c}.$$

& $\int_0^1 (1 - 2\sqrt{ac}) da = 1 - \frac{4\sqrt{c}}{3}.$

$$\int_{1/4}^1 \int_0^{\frac{1}{4c}} (1 - 2\sqrt{ac}) da \, dc = \int_{1/4}^1 \frac{1}{12c} \, dc = \frac{1}{12} \ln 4 = \frac{\ln 2}{6}.$$

$$\int_0^{1/4} \int_0^1 (1 - 2\sqrt{ac}) da \, dc = \int_0^{1/4} \left(1 - \frac{4\sqrt{c}}{3}\right) dc$$

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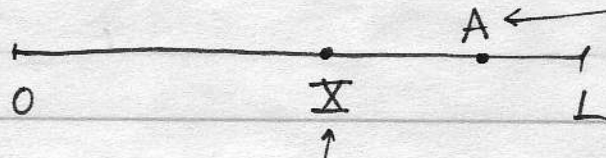

A ← place of ambulance = unif $(0, L)$

place of accident = unif $(0,L)$ r.v.

Want $$E\left(|X-A|\right) = \int_0^L \int_0^L |x-a|\, f(x,a)\, dx\, da,$$

where $$f(x,a) = f_X(x)\, f_L(a)$$
$$= \begin{cases} 1/L^2 & \text{if } 0 \le x, a \le L, \\ 0 & \text{o/w}. \end{cases}$$

$$\Rightarrow \quad E(|X-a|) = \int_0^L \int_0^L |x-a| / L^2 \; dx\, da$$
$$= \int_0^L \int_0^a \frac{a-x}{L^2}\, dx\, da + \int_0^L \int_a^L \frac{x-a}{L^2}\, dx\, da.$$

$$\int_0^a \frac{a-x}{L^2}\, dx = \frac{1}{2}\frac{a^2}{L^2}.$$

$$\int_a^L \frac{x-a}{L^2}\, dx = \frac{1}{2}\frac{(L-a)^2}{L^2}.$$

$\Rightarrow$

$$E(|X-a|) = \frac{1}{2L^2}\left(\int_0^L a^2\, da + \int_0^L (L-a)^2\, da\right) = \frac{L}{3}.$$

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$$= \frac{1}{4} - \frac{4}{3}\int_0^{1/4} c^{\frac{1}{2}}\,dc$$

$$= \frac{1}{4} - \frac{4}{3}\,\frac{2}{3}\left(\frac{1}{4}\right)^{3/2} = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}.$$

$$\Rightarrow \quad P\{\text{roots real}\} = \frac{5}{36} + \frac{\ln 2}{6} \simeq 0.2544.$$

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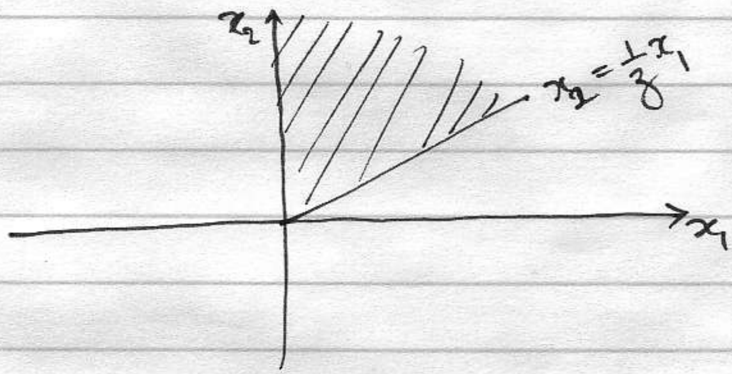
$$F_Z(z) = P\{X_1 \le z X_2\} = P\{(X_1, X_2) \text{ is in the shaded area}\}.$$



$(z > 0)$

$$= \int_0^\infty \int_0^{z x_2} \lambda_1 e^{-\lambda_1 x_1}\,\lambda_2 e^{-\lambda_2 x_2}\,dx_1\,dx_2$$

$$= \lambda_1\lambda_2 \int_0^\infty e^{-\lambda_2 x_2} \underbrace{\left(\int_0^{z x_2} e^{-\lambda_1 x_1}\,dx_1\right)}_{\frac{1}{\lambda_1}\left(1 - e^{-\lambda_1 z x_2}\right)} dx_1$$

$$= \lambda_2 \int_0^\infty \left[e^{-\lambda_2 x_2} - e^{-(\lambda_2 + z\lambda_1)x_2}\right] dx_2$$

$$= 1 - \frac{\lambda_2}{\lambda_2 + z\lambda_1} = \frac{z\,\lambda_1}{\lambda_2 + z\lambda_1}.$$

$F_Z(z) = 0 \quad$ if $z < 0$.

$$\therefore \quad f_Z(z) = F_Z'(z)$$

$$= \frac{\lambda_1 (x_2 + z\lambda_1) - z\lambda_1 (\lambda_1)}{(\lambda_2 + z\lambda_1)^2}$$

$$= \frac{\lambda_1 \lambda_2}{(\lambda_2 + z\lambda_1)^2}, \quad \text{if } z > 0.$$

If $z \le 0$ then $f_Z(z) = 0$.

$$P\{X_1 < X_2\} = F_Z(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

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ⓐ $P(X_1 = x \mid X_2 = 1) = ?$

$$p_{X_1|X_2}(0|1) = \frac{P(X_1 = 0, X_2 = 1)}{P(X_2 = 1)} = \frac{P(X_2 = 1 \mid X_1 = 0)\, P(X_1 = 0)}{P(X_2 = 1)}$$

$$= \frac{(5/12) \times (8/13)}{(5/13)} = \frac{2}{3}.$$

$$p_{X_1|X_2}(1|1) = 1 - p_{X_1|X_2}(0|1) = 1/3.$$

ⓑ $$p_{X_1|X_2}(0|0) = \frac{P(X_2 = 0 \mid X_1 = 0)\, P(X_1 = 0)}{P(X_2 = 0)} = \frac{(7/12)(8/13)}{8/13} = \frac{7}{12}$$

$$p_{X_1|X_2}(1|0) = 1 - \frac{7}{12} = \frac{5}{12}.$$

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Th. problem #5, p319

ⓐ $F_Z(z) = P(X \le zY) = \int_0^\infty \int_0^{zy} f(x,y)\,dx\,dy$ (see #28)

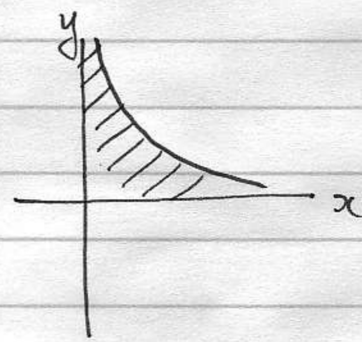
$$= \int_0^\infty \int_0^{zy} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_0^\infty f_Y(y) F_X(zy)\,dy.$$

$$f_Z(y) = F_Z'(z) = \int_0^\infty f_Y(y) F_X'(zy)\,y\,dy$$

$$= \int_0^\infty f_Y(y) f_X(zy)\,y\,dy.$$

ⓑ $F_Z(z) = P\{Y \le z/X\} =$



$$= \int_0^\infty \int_0^{z/y} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_0^\infty F_X(z/y) f_Y(y)\,dy.$$

$$f_Z(z) = \int_0^\infty f_X(z/y) f_Y(y) \frac{1}{y}\,dy.$$

If $f_X(x) = \lambda_1 e^{-\lambda_2 x}$ $(x>0)$ $\quad f_Y(y) = \lambda_2 e^{-\lambda_2 y}$ $(y>0)$
then

$$f_{X/Y}(z) = \int_0^\infty \lambda_2 e^{-\lambda_2 y} \lambda_1 e^{-\lambda_1 z y}\, y\,dy$$

$$= \lambda_1 \lambda_2 \int_0^\infty y\, e^{-(\lambda_1 z + \lambda_2) y}\,dy$$

$$= \frac{\lambda_1\lambda_2}{(\lambda_1 z + \lambda_2)^2} \underbrace{\Gamma(2)}_{1} = \frac{\lambda_1\lambda_2}{(\lambda_1 z + \lambda_2)^2}.$$ [as in #28]

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And $f_{XY}(z) = \int_0^\infty \lambda_1 e^{-\lambda_1 z/y} \lambda_2 e^{-\lambda_2 y} \frac{1}{y}\, dy$

$$= \lambda_1\lambda_2 \int_0^\infty e^{-\lambda_1 z/y - \lambda_2 y} \frac{1}{y}\, dy.$$

This can be simplified (a little): let $y = \sqrt{\frac{\lambda_1 z}{\lambda_2}}\, u$ to see that

$$f_{XY}(z) = \lambda_1\lambda_2 \int_0^\infty e^{-\sqrt{\lambda_1 z/\lambda_2}\left[\frac{1}{u}+u\right]} \frac{du}{u}.$$