Assignment 6 - Math 5010 - Summer 2006

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(a) $P\{|X| > \frac{1}{2}\} = P\{X > \frac{1}{2}\} + P\{X < -\frac{1}{2}\}$

$$= \int_{\frac{1}{2}}^{1} \frac{1}{2}\,dx + \int_{-1}^{-\frac{1}{2}} \frac{1}{2}\,dx = 1/2$$

(b) $P\{|X| \le x\} = \int_{-x}^{x} \frac{1}{2}\,dy$ if $0 \le x \le 1$

$= x = F_{|X|}(x)$. Else $F_{|X|}(x) = 0$.

$$\Rightarrow f_{|X|}(x) = F_{|X|}(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & o/w \end{cases}$$

$\Rightarrow |X|$ is Unif $(0,1)$.

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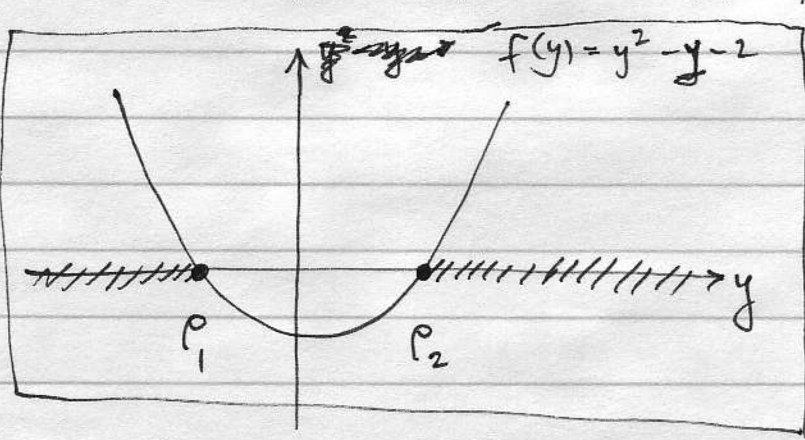
Roots real if and only if $(4Y)^2 - 4(Y+2)(4) \geq 0$

$\leftrightarrow \quad 16Y^2 - 16Y - 32 \geq 0$

$\leftrightarrow \quad Y^2 - Y - 2 \geq 0 \quad \leftrightarrow$ $Y$ is in the shaded region below




Need $\rho_1$ and $\rho_2$:

$$\rho_1 = \frac{1 - \sqrt{1+8}}{2} = \frac{1-3}{2} = -1$$

$$\rho_2 = \frac{1+3}{2} = 2$$

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$Y \in [0,5] \Rightarrow \quad P\{\text{roots real}\} = P\{2 \le Y \le 5\}$

$$= \int_2^5 \frac{1}{5}\,dy = 3/5.$$

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$$F_Y(y) = P\{\log X \le y\}$$

$$= P\{X \le e^y\}$$

$$= \int_0^{e^y} e^{-x}\,dx = 1 - e^{-e^y}, \qquad -\infty < y < \infty.$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y) = e^{-e^y} \times e^y$$

$$= \exp\left(y - e^y\right), \qquad -\infty < y < \infty.$$

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$$F_Y(y) = P\{e^X \le y\} = P\{X \le \ln y\} \qquad y \ge 1$$

$$= \begin{cases} 1 & y > e \\ \ln y & 1 < y < e \\ 0 & \text{o/w} \end{cases}$$

$$\therefore \quad f_Y(y) = \begin{cases} 1/y & 1 < y < e \\ 0 & \text{o/w} \end{cases}$$

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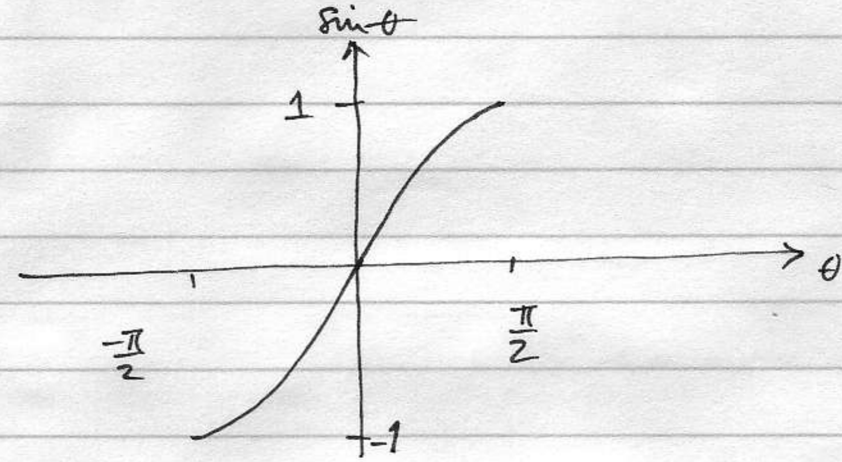
We assume that $A > 0$. Then,

$$F_R(r) = P\{A \sin \Theta \leq r\} = P\left\{\sin \Theta \leq \frac{r}{A}\right\}$$

$$= \begin{cases} 1 & r > A \\ P\{\Theta \leq \sin^{-1}(r/A)\} & -A < r < A \\ 0 & r \leq A \end{cases}$$




If $-A < r < A$ then $\quad F_R(r) = \displaystyle\int_{-\frac{\pi}{2}}^{\sin^{-1}(r/A)} \frac{1}{\pi}\, d\theta$

$$= \frac{1}{\pi}\left[\sin^{-1}\left(\frac{r}{A}\right) + \frac{\pi}{2}\right]$$

$$\frac{d}{dr} \sin^{-1}\left(\frac{r}{A}\right) = \frac{1}{A} \frac{1}{\sqrt{1 - \left(\frac{r}{A}\right)^2}} = \frac{1}{\sqrt{A^2 - r^2}}.$$

$$\Rightarrow \quad f_R(r) = \begin{cases} \dfrac{1}{\pi\sqrt{A^2 - r^2}} & |r| \leq A \\ 0 & \text{o/w} \end{cases}$$

Assignment 5

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$X$ Weibull$(\nu, \alpha, \beta)$ means: $[\cancel{\nu}, \alpha, \beta > 0 \ , \ \nu \in \mathbb{R}]$

$$f_X(x) = \begin{cases} \dfrac{\beta}{\alpha}\left(\dfrac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\dfrac{x-\nu}{\alpha}\right)^{\beta}\right\} & x > \nu \\ 0 & \text{o/w} \end{cases}$$

(§5.6.2, page 239).

Now $F_Y(y) = P\left\{\left(\dfrac{X-\nu}{\alpha}\right)^{\beta} \le y\right\}$ $\qquad y > 0$

$$= P\left\{\frac{X-\nu}{\alpha} \le y^{1/\beta}\right\}$$

$$= P\left\{X \le \alpha y^{1/\beta} + \nu\right\}$$

$$= F_X(\alpha y^{1/\beta} + \nu).$$ Else $F_Y(y) = 0$ for $y < 0$.

So $$f_Y(y) = F_X'(\overbrace{\alpha y^{1/\beta} + \nu}^{x}) \cdot \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1}$$

$$= \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left(-\left[\frac{x-\nu}{\alpha}\right]^{\beta}\right) \frac{\alpha}{\beta} y^{1/\beta - 1}$$

$$= \frac{\beta}{\alpha}\left(y^{1/\beta}\right)^{\beta-1} \exp\left(-y\right) = e^{-y} \qquad y > 0.$$

$f_Y(y) = 0$ for $y < 0$.

$\Rightarrow$ $Y$ is $\mathcal{E}xp(1)$.

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Recall $X$ : Beta$(a,b)$ means (§5.6.4) [$a,b>0$]

$$f_X(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} & 0 \le x \le 1, \\ 0 & \text{o/w}. \end{cases}$$

Here, $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ is the so called "Beta function"

$$EX = \int_0^1 \frac{1}{B(a,b)} x^{a} (1-x)^{b-1}\,dx = \frac{B(a+1,b)}{B(a,b)}$$

$$EX^2 = \int_0^1 \frac{1}{B(a,b)} x^{a+1} (1-x)^{b-1}\,dx = \frac{B(a+2,b)}{B(a,b)}.$$

Fact (cf. (6.3) on p. 241) : $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

$$\Rightarrow \quad EX = \frac{\Gamma(a+1)\Gamma(b)/\Gamma(a+1+b)}{\Gamma(a)\Gamma(b)/\Gamma(a+b)}$$

$$= \frac{\Gamma(a+1)}{\Gamma(a)} \cdot \frac{\Gamma(a+b)}{\Gamma(a+1+b)} = a \cdot \frac{1}{a+b} = \frac{a}{a+b}.$$

$$EX^2 = \frac{\Gamma(a+2)\Gamma(b)/\Gamma(a+b+2)}{\Gamma(a)\Gamma(b)/\Gamma(a+b)}$$

$$= \frac{\Gamma(a+2)}{\Gamma(a)} \cdot \frac{\Gamma(a+b)}{\Gamma(a+b+2)} = \frac{(a+1)\Gamma(a+1)}{\Gamma(a)} \cdot \frac{\Gamma(a+b)}{(a+b+1)\Gamma(a+b+1)}$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)}.$$

$\mathrm{Var}\,X = EX^2 - (EX)^2$. The rest is a little bit of algebra.