Math 5010-1

Solution to Assignment 5

#5. Let c = desired tank capacity

X = volume of weekly sales.

Want

$$P\{X \geq c\} = 0.01.$$

But

$$P\{X \geq c\} = \int_c^1 5(1-x)^4\,dx \qquad (\text{if } 0<c<1)$$

$$= \int_0^{1-c} 5y^4\,dy \qquad (y = 1-x)$$

$$= (1-c)^5.$$

Set $(1-c)^5 = 0.01 \Rightarrow 1-c = 0.01^{1/5}$

$$\Rightarrow \boxed{c = 1 - 0.01^{1/5}}$$

#6. (a) $$EX = \int_0^\infty \frac{x^2}{4} e^{-x/4}\,dx = 16\int_0^\infty y^2 e^{-y}\,dy \qquad \left(y = \frac{x}{4}\right)$$

$$= 16\,\Gamma(3) = 16 \times 2! = 32$$

(b) First find c:

$$\int_{-1}^1 c(1-x^2)\,dx = 1 \Rightarrow$$

$$c = \frac{1}{\int_{-1}^1 (1-x^2)\,dx} = \frac{1}{2 - \frac{2}{3}} = 3/4.$$

p-2
Math 5010
Assign. #5

$$\Rightarrow \quad f(x) = \begin{cases} \frac{3}{4}(1-x^2) & \text{if } -1 \le x \le 1, \\ 0 & \text{o/w}. \end{cases}$$

$$\Rightarrow \quad EX = \int_{-1}^{1} \frac{3}{4} x(1-x^2)\,dx$$

$$= \frac{3}{4}\int_{-1}^{1} x\,dx - \frac{3}{4}\int_{-1}^{1} x^3\,dx$$

$$= 0 \qquad \text{[or use symmetry directly]}$$

(c) $$f(x) = \begin{cases} 5/x^2 & x > 5 \\ 0 & \text{o/w} \end{cases}$$

$$\Rightarrow \quad EX = 5\int_5^{\infty} \frac{1}{x}\,dx = \infty.$$

#13. Let X = your wait. Then X is unif $(0,30)$ [in min].

(a) $$P\{X \geqslant 10\} = \int_{10}^{30} \frac{1}{30}\,dx = 2/3.$$

(b) $$P\{X \geqslant 25 \mid X \geqslant 15\} = P\{X \geqslant 25, X \geqslant 15\} \Big/ P\{X \geqslant 15\}$$

$$= \frac{P\{X \geqslant 25\}}{P\{X \geqslant 15\}} = \frac{\int_{25}^{30} \frac{1}{30}\,dx}{\int_{15}^{30} \frac{1}{30}\,dx} = 1/3.$$

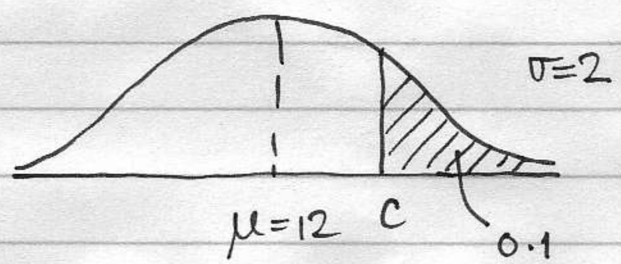
p-3
Math 5010
Assign. #5

#17. Let X = his reward. Then

$$X = \begin{cases} 10 & \text{with probab } \frac{1}{10} \\ 5 & \text{" } \text{" } 2/10 \\ 3 & \text{" } \text{" } 2/10 \\ 0 & \text{" } \text{" } 5/10 \end{cases}$$

$$\Rightarrow \quad EX = 10\times\frac{1}{10} + 5\times\frac{2}{10} + 3\times\frac{2}{10} = 2.6 \text{ \$}.$$

#19.



$$0.1 = P\{X \geq c\} = P\left\{\frac{X-\mu}{\sigma} \geq \frac{c-12}{2}\right\} = 1 - \Phi\left(\frac{c-12}{2}\right).$$

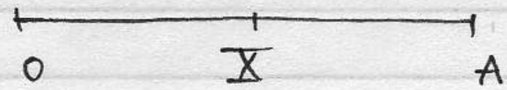
$$\Rightarrow \quad \Phi\left(\frac{c-12}{2}\right) = 0.9$$

$$\Rightarrow \quad 1.28 \leq \frac{c-12}{2} \leq 1.29$$

$$\Rightarrow \quad 12 + (2\times 1.28) \leq c \leq 12 + (2\times 1.29)$$

$$14.56 \leq c \leq 14.58.$$

#31. (a)



$$E(|X-a|) = \int_0^A |x-a| \frac{1}{A}\,dx$$

$$= \int_0^a (a-x)\frac{1}{A}\,dx \qquad \left[\text{if } 0<a<A\right]$$

$$+ \int_a^A (x-a)\frac{1}{A}\,dx$$

$$= \frac{1}{2}\frac{a^2}{A} + \frac{1}{2}\frac{(A-a)^2}{A}.$$

$\Rightarrow \min\limits_{0<a<A} E(|X-a|)$ is the same as

$$\min_{0<a<A}\left[a^2 + (A-a)^2\right] \equiv \min_{0<a<A} h(a).$$

$$h'(a) = 2a - 2(A-a) = 4a - 2A$$

$$h''(a) = 4 \geq 0 \longrightarrow \text{min at } a = A/2.$$

<u>Question</u> What if you consider also $a \notin [0,A]$?

(b) Now, $E(|X-a|) = \int_0^a (a-x)\lambda e^{-\lambda x}\,dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}\,dx$

$$= \int_0^a y\lambda e^{-\lambda(a-y)}\,dy + \int_0^\infty y\lambda e^{-\lambda(y+a)}\,dy$$

$$= \lambda e^{-\lambda a}\int_0^a y e^{\lambda y}\,dy + \lambda e^{-\lambda a}\int_0^\infty y e^{-\lambda y}\,dy.$$

Now $\int_0^\infty y e^{-\lambda y}\, dy = \frac{1}{\lambda^2} \int_0^\infty z e^{-z}\, dz = \frac{\Gamma(2)}{\lambda^2} = \frac{1}{\lambda^2}$.

Also,

$$\int_0^a y e^{\lambda y}\, dy = \frac{1}{\lambda} e^{\lambda y} y \Big|_0^a - \int_0^a \frac{1}{\lambda} e^{\lambda y}\, dy$$

$$= \frac{1}{\lambda} a e^{\lambda a} + \frac{1}{\lambda^2}\left(1 - e^{\lambda a}\right).$$

$$\Rightarrow \quad E(|X - a|) = \lambda e^{-\lambda a}\left[\frac{1}{\lambda} a e^{\lambda a} + \frac{1}{\lambda^2}\left[1 - e^{\lambda a}\right]\right]$$

$$+ \lambda e^{-\lambda a} \cdot \frac{1}{\lambda^2}$$

$$= a + \frac{1}{\lambda} e^{-\lambda a} - \frac{1}{\lambda} + \frac{1}{\lambda} e^{-\lambda a}$$

$$= a + \frac{2}{\lambda} e^{-\lambda a} - \frac{1}{\lambda}.$$

$$\equiv h(a).$$

$h'(a) = -2e^{-\lambda a} + 1$

$h''(a) = 2\lambda e^{-\lambda a} \geq 0 \Rightarrow$ min at $h'(a) = 0$; but this is

$0 = h'(a) = -2e^{-\lambda a} + 1 \Rightarrow e^{-\lambda a} = 1/2$

$\Rightarrow \quad a = \frac{1}{\lambda} \ln 2.$

p-6
Math 5010
Assign #5

Th. Problem #12.

(a) If $a \le x \le b$ then

$$F(x) = \int_a^x \frac{1}{b-a}\,dy = \frac{x-a}{b-a}.$$

Set this $= \frac{1}{2}$ $\Rightarrow$ $m = a + \frac{1}{2}(b-a) = \frac{b+a}{2}$.

(b) By symmetry, $m = \mu$.

(c) $F(x) = \int_0^x \lambda e^{-\lambda y}\,dy$ (if $x \geq 0$)

$$= 1 - e^{-\lambda x}.$$

Set this $= \frac{1}{2}$ $\Rightarrow$ $e^{-\lambda m} = \frac{1}{2}$

$$\Rightarrow \quad m = \frac{1}{\lambda}\ln 2.$$

Compare with problem 31(b) to see a "pattern."