

Solutions to Homework #4
Mathematics 5010–1, Summer 2006
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Read this only after you have really thought hard about the problems.

Problems:

#8, p. 187. (a) We have

$$p(k) = \begin{cases} 1/36, & \text{if } k = 1, \\ 3/36, & \text{if } k = 2, \\ 5/36, & \text{if } k = 3, \\ 7/36, & \text{if } k = 4, \\ 9/36, & \text{if } k = 5, \\ 11/36, & \text{if } k = 6, \\ 0, & \text{otherwise.} \end{cases}$$

[For instance, $p(2)$ is the probability that you roll either a $(2, 1)$, $(1, 2)$, or a $(2, 2)$.]

(b) We have

$$p(k) = \begin{cases} 11/36, & \text{if } k = 1, \\ 9/36, & \text{if } k = 2, \\ 7/36, & \text{if } k = 3, \\ 5/36, & \text{if } k = 4, \\ 3/36, & \text{if } k = 5, \\ 1/36, & \text{if } k = 6, \\ 0, & \text{otherwise.} \end{cases}$$

[For instance, $p(1)$ is the probability that you roll either a $(1, 1)$, $(1, 2)$, $(2, 1)$, $\dots$, or a $(6, 1)$.]

(c) We have

$$p(k) = \begin{cases} 1/36, & \text{if } k = 2, \\ 2/36, & \text{if } k = 3, \\ 3/36, & \text{if } k = 4, \\ 4/36, & \text{if } k = 5, \\ 5/36, & \text{if } k = 6, \\ 6/36, & \text{if } k = 7, \\ 5/36, & \text{if } k = 8, \\ 4/36, & \text{if } k = 9, \\ 3/36, & \text{if } k = 10, \\ 2/36, & \text{if } k = 11, \\ 1/36, & \text{if } k = 12, \\ 0, & \text{otherwise.} \end{cases}$$

[For instance, $p(5)$ is the probability that you roll either a $(1, 4)$, $(4, 1)$, $(2, 3)$, or a $(3, 2)$.]

(d) We have

$$p(k) = \begin{cases} 1/36, & \text{if } k = -5, \\ 2/36, & \text{if } k = -4, \\ 3/36, & \text{if } k = -3, \\ 4/36, & \text{if } k = -2, \\ 5/36, & \text{if } k = -1, \\ 6/36, & \text{if } k = 0, \\ 5/36, & \text{if } k = 1, \\ 4/36, & \text{if } k = 2, \\ 3/36, & \text{if } k = 3, \\ 2/36, & \text{if } k = 4, \\ 1/36, & \text{if } k = 5, \\ 0, & \text{otherwise.} \end{cases}$$

[For instance, $p(-2)$ is the probability that you roll either a $(1, 3)$, $(2, 4)$, $(3, 5)$, or a $(4, 6)$.]

#13, p. 187. The possible values for X are: \$0; \$500; \$1000; \$1500; or \$2000. The mass function is

$$p(k) = \begin{cases} 0.4 \times 0.3 & \text{if } k = 0, \\ \{(0.3 \times 0.5) \times 0.4\} + \{0.7 \times (0.4 \times 0.5)\}, & \text{if } k = 500, \\ \{(0.3 \times 0.5) \times (0.4 \times 0.5)\} + \{(0.3 \times 0.5) \times 0.4\} + \{(0.7 \times (0.6 \times 0.5))\}, & \text{if } k = 1000, \\ 2 \times \{(0.3 \times 0.5) \times (0.6 \times 0.5)\}, & \text{if } k = 1500, \\ \{(0.3 \times 0.5) \times (0.6 \times 0.5)\}, & \text{if } k = 2000. \end{cases}$$

[For instance, here is how you can compute $p(1500)$: Let $D_i = \{\text{deluxe on the } i\text{th sale}\}$, and $S_i = \{\text{sale on the } i\text{th}\}$. Then $p(1500) = P\{(D_1 \cap S_1) \cap (D_1^c \cap S_2)\} + P\{(D_1^c \cap S_1) \cap (D_2 \cap S_2)\}$.]

#22, p. 190. Let X denote the total number of games played. We are asked to compute EX .

- (a) Suppose $i = 2$. Write AB for the event that “Team A won the first round, and Team B the second.” Likewise, write ABA , etc. We can compute the mass function of X by first noting that the possible values of X are: 2 [this happens if and only if $AA \cup BB$ happens] and 3 [$ABA \cup BAB \cup ABB \cup BAA$]. The mass function is:

$$p(k) = \begin{cases} p^2 + (1-p)^2 = 2p^2 - 2p + 1, & \text{if } k = 2, \\ 2p^2(1-p) + 2(1-p)^2p = 2p(1-p), & \text{if } k = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$EX = 2[2p^2 - 2p + 1] + 3[2p(1-p)] = 2 + 2p - 2p^2.$$

Let $h(p) := 2 + 2p - 2p^2$ and note that: (i) $h'(p) = 2 - 4p$; and (ii) $h''(p) = -4$. Because $h''(p) < 0$, we solve the problem by setting $h'(p) \equiv 0$. This yields, $p_{\max} = \boxed{1/2}$.

- (b) When $i = 3$ the possible values of X are: 3 [$AAA \cup BBB$]; 4 [$AABA \cup ABAA \cup BBAB \cup BABB$]; or 5 [$AABBA \cup ABBAA \cup ABABA \cup BBAAB \cup BABAB \cup BAABB$]. The mass function is:

$$p(k) = \begin{cases} p^3 + (1-p)^3, & \text{if } k = 3, \\ 2p^3(1-p) + 2p(1-p)^3, & \text{if } k = 4, \\ 3p^3(1-p)^2 + 3p^2(1-p)^3, & \text{if } k = 5, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} EX &= 3(p^3 + (1-p)^3) + 4(2p^3(1-p) + 2p(1-p)^3) + 5(3p^3(1-p)^2 + 3p^2(1-p)^3) \\ &= F(p) + F(1-p), \end{aligned}$$

where $F(p) := 3p^3 + 8p^3(1-p) + 15p^3(1-p)^2$. Let $h(p) = EX$, so that $h(p) = F(p) + F(1-p)$. Then,

$$\begin{aligned} h'(p) &= F'(p) - F'(1-p), \\ h''(p) &= F''(p) + F''(1-p). \end{aligned}$$

Now,

$$\begin{aligned} F'(p) &= 9p^2 + 24p^2(1-p) - 8p^3 + 45p^2(1-p)^2 - 30p^3(1-p), \\ F''(p) &= 18p^2 + 48p(1-p) - 24p - 24p + 90p(1-p)^2 - 90p^2(1-p) \\ &= -40p^2 + 90p(1-p)^2 - 90p^2(1-p). \end{aligned}$$

Therefore,

$$\begin{aligned} F''(p) + F''(1-p) &= -40p^2 + 90p(1-p)^2 - 90p^2(1-p) - 40(1-p)^2 + 90(1-p)p^2 - 90(1-p)^2p \\ &= -40[p^2 + (1-p)^2] < 0. \end{aligned}$$

Therefore, it suffices to solve for $h'(p) = 0$; i.e., find p such that $F'(p) = F'(1-p)$. Obviously, $p = 1/2$ works.

#28, p. 191. Let X denote the total number of defectives in the sample. Then, X has the hypergeometric distribution (§4.8.3) with parameters $N = 20$, $n = 3$, and $m = 4$. According to Example 8j (see p. 181), $EX = nm/N = \boxed{3/5}$.

#31, p. 192. Let X denote the score of the said meteorologist. Then,

$$EX = p^* [1 - (1 - p)^2] + (1 - p^*) [1 - p^2].$$

Call this $h(p)$. Then,

$$\begin{aligned} h'(p) &= 2p^*(1 - p) - 2p(1 - p^*) = 2p^* - 2p, \\ h''(p) &= -2. \end{aligned}$$

Because $h'' < 0$ we need only solve $h' \equiv 0$. That is, $\boxed{p = p^*}$: It is best to be truthful in this case.

#35, p. 192. Let X denote the total win, and note that $P\{X = 1.1\} = 4/9$ and $P\{X = -1\} = 5/9$ (why?).

(a) $EX = \frac{4}{9} \times 1.1 + \frac{5}{9} \times (-1) = \boxed{0.01} = -0.06\bar{6} \approx -7\text{¢}$.

(b) First, $E(X^2) = 1.1^2 \times \frac{4}{9} + (-1)^2 \times \frac{5}{9} = 1.09\bar{3}$. Then, $\text{Var}(X) = 1.105 - 0.01^2 \approx \boxed{\$1.0885}$.

Th. Exercises:

10, p. 198 I will show two proofs: The first is what I think you are supposed to do; it is very close to the computations that we have made several times. The second proof is shorter, but more tricky.

Proof 1. Evidently,

$$\begin{aligned} E \left[\frac{1}{1+X} \right] &= \sum_{k=0}^n \frac{1}{1+k} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{n=0}^k \frac{n!}{(k+1)!(n-k)!} p^k (1-p)^{n-k} \\ &= \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k+1} p^k (1-p)^{n-k} = \frac{1}{(n+1)p} \sum_{k=0}^n \binom{n+1}{k+1} p^{k+1} (1-p)^{(n+1)-(k+1)} \\ &= \frac{1}{(n+1)p} \sum_{j=1}^{n+1} \binom{n+1}{j} p^j (1-p)^{n+1-j} = \\ &= \frac{1}{(n+1)p} \left[\sum_{j=0}^{n+1} \binom{n+1}{j} p^j (1-p)^{n+1-j} - (1-p)^{n+1} \right], \end{aligned}$$

because $(1-p)^{n+1}$ is the zeroth term in the sum. The latter sum is one, thanks to the binomial theorem. Therefore, the assertion follows.

Proof 2. We know that

$$\frac{1}{1+k} = \int_0^\infty e^{-t(1+k)} dt.$$

Therefore,

$$E \left[\frac{1}{1+X} \right] = \int_0^\infty \underbrace{\sum_{k=0}^n \binom{n}{k} (e^{-tp})^k (1-p)^{n-k}}_{:=Q(t)} e^{-t} dt$$

By the binomial theorem,

$$Q(t) = (e^{-t}p + (1-p))^n.$$

Therefore,

$$\begin{aligned} E\left[\frac{1}{1+X}\right] &= \int_0^\infty Q(t)e^{-t} dt = \int_0^\infty (e^{-t}p + 1-p)^n e^{-t} dt \\ &= \int_0^1 (sp + 1-p)^n ds \quad [s := e^{-t}]. \end{aligned}$$

Now integrate to finish.

#17, p. 198(b). [Challenge problem] By definition,

$$P\{X \text{ is even}\} = e^{-\lambda} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{\lambda^n}{n!}.$$

According to Taylor's expansion,

$$e^\lambda + e^{-\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} = 2 \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{\lambda^n}{n!}.$$

[Perform one expansion for $e^{-\lambda}$ and another for e^λ .] Therefore,

$$P\{X \text{ is even}\} = \frac{e^{-\lambda}}{2} (e^\lambda + e^{-\lambda}) = \frac{1}{2} [1 + e^{-2\lambda}],$$

as asserted. Here is a hint for part **(a)**: First compute $P\{B_n \text{ is even}\}$, where B_n is binomial($n, \frac{1}{n}$). But be warned that this part needs some theoretical developments that we have not yet discussed during lecture.