Math 5010–1, Exam 4

July 30, 2013

- Your name must read clearly and legibly on every sheet top.
- Please clarify which problem is solved on every unmarked sheet.
- You may use a calculator and notes, if you wish.
- Show your work in detail. No credit is given for writing an answer [even if it happens to be correct.]
- This exam ends in one hour; there are 4 questions.

1. (10 points) Let X and Y be two independent standard normal random variables. That is, they have the common pdf,

$$f(a) = \frac{e^{-a^2/2}}{\sqrt{2\pi}} \qquad (-\infty < a < \infty).$$

Let $(R\,,\Theta)$ be the polar coordinates description of $(X\,,Y)$. That is, $X = R\cos\Theta$ and $Y = R\sin\Theta$. Prove that R and Θ are independent, and compute their respective marginal pdf's.

Solution. The Jacobian is

$$\det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = r.$$

Therefore,

$$f_{R,\Theta}(r\,,\theta) = \frac{1}{2\pi} r\mathrm{e}^{-r^2/2} \qquad (0 < \theta < 2\pi,\ r > 0).$$

This means that R and Θ are independent and

$$f_R(r) = r\mathrm{e}^{-r^2/2} I\{x > 0\}, \qquad f_\Theta(\theta) = \frac{1}{2\pi} I\{0 < \theta < 2\pi\}.$$

2. (25 points total) Let X be a random variable with the following pdf:

$$f(a) = \begin{cases} ae^{-a^2/2} & \text{if } a > 0,\\ 0 & \text{otherwise.}\end{cases}$$

(a) (5 points) Verify that f is indeed a probability density function.

(b) (10 points) Compute $\mathrm{E}(X)$.

Solution.

$$\mathrm{E}(X) = \int_0^\infty a^2 e^{-a^2/2}\, da = \sqrt{2}\int_0^\infty y^{1/2} e^{-y}\, dy = \sqrt{2}\Gamma(3/2) = \sqrt{\pi/2}$$

(c) (10 points) Compute $\mathrm{Var}(X)$.

Solution. First we compute

$$\mathrm{E}(X^2) = \int_0^\infty a^3 e^{-a^2/2}\, da = 2\int_0^\infty y e^{-y} dy = 2\Gamma(2) = 2.$$

Therefore, $\mathrm{Var}(X) = 2 - (\pi/2)$.

3. (10 points) Suppose X and Y are independent random variables, both distributed uniformly on $[-1\,,1]$. Compute $\mathrm{E}[\max(X\,,Y)]$.

Solution. First of all,

$$F_X(a) = F_Y(a) = \begin{cases} 1 & \text{if } a > 1, \\ \dfrac{a+1}{2} & \text{if } -1 < a < 1, \\ 0 & \text{if } a < -1. \end{cases}$$

Let $Z := \max(X\,,Y)$ and notice that

$$F_Z(a) = \mathrm{P}\{X \le a\}\mathrm{P}\{Y \le a\} = \begin{cases} 1 & \text{if } a > 1, \\ \dfrac{(a+1)^2}{4} & \text{if } -1 < a < 1, \\ 0 & \text{if } a < -1. \end{cases}$$

Therefore,

$$f_Z(a) = \begin{cases} \dfrac{a+1}{2} & \text{if } -1 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$\mathrm{E}\max(X\,,Y) = \mathrm{E}Z = \int_{-1}^1 a\left(\frac{a+1}{2}\right) da = \frac{1}{3}.$$

4. (10 points) What is the expected value of X when X has the following probability mass function:

$$\mathrm{P}\{X = a\} = \begin{cases} \dfrac{1}{4}\left(\dfrac{3}{4}\right)^{a-1} & \text{if } a = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Let $p = {}^1\!/_4$ and $q := {}^3\!/_4$, so that $f(a) = pq^{a-1}I\{a = 1, 2, \cdots\}$. Then,

$$\mathrm{E}(X) = p\sum_{a=1}^{\infty} aq^{a-1} = p\frac{d}{dq}\sum_{a=0}^{\infty} q^a = p\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{1}{p} = 4.$$