1. Two important properties

**Theorem 27.1** (Uniqueness). If $X$ and $Y$ are two random variables—discrete or continuous—with moment generating functions $M_X$ and $M_Y$, and if there exists $\delta > 0$ such that $M_X(t) = M_Y(t)$ for all $t \in (-\delta, \delta)$, then $M_X = M_Y$ and $X$ and $Y$ have the same distribution. More precisely:

1. $X$ is discrete if and only if $Y$ is, in which case their mass functions are the same;
2. $X$ is continuous if and only if $Y$ is, in which case their density functions are the same.

**Theorem 27.2** (Lévy’s continuity theorem). Let $X_n$ be a random variables—discrete or continuous—with moment generating functions $M_n$. Also, let $X$ be a random variable with moment generating function $M$. Suppose there exists $\delta > 0$ such that:

1. If $-\delta < t < \delta$, then $M_n(t), M(t) < \infty$ for all $n \geq 1$; and
2. $\lim_{n \to \infty} M_n(t) = M(t)$ for all $t \in (-\delta, \delta)$, then

$$\lim_{n \to \infty} F_{X_n}(a) = \lim_{n \to \infty} P\{X_n \leq a\} = P\{X \leq a\} = F_X(a),$$

for all numbers $a$ at which $F_X$ is continuous.

**Example 27.3** (Law of rare events). Suppose $X_n = \text{binom}(n, \lambda/n)$, where $\lambda > 0$ is fixed, and $n \geq \lambda$. Then, recall that

$$M_{X_n}(t) = (q + pe^{-t})^n = \left(1 - \frac{\lambda}{n} + \frac{\lambda e^{-t}}{n}\right)^n \to \exp(-\lambda + \lambda e^{-t}).$$
Note that the right-most term is \( M_X(t) \), where \( X = \text{Poisson}(\lambda) \). Therefore, by Lévy’s continuity theorem,

\[
\lim_{n \to \infty} \Pr \{ X_n \leq a \} = \Pr \{ X \leq a \},
\]

(20)

at all \( a \) where \( F_X \) is continuous. But \( X \) is discrete and integer-valued. Therefore, \( F_X \) is continuous at \( a \) if and only if \( a \) is not a nonnegative integer.

If \( a \) is a nonnegative integer, then we can choose a non-integer \( b \in (a, a+1) \) to find that

\[
\lim_{n \to \infty} \Pr \{ X_n \leq b \} = \Pr \{ X \leq b \}.
\]

Because \( X_n \) and \( X \) are both non-negative integers, \( X_n \leq b \) if and only if \( X_n \leq a, \) and \( X \leq b \) if and only if \( X \leq a. \) Therefore, (20) holds for all \( a. \)

**Example 27.4** (The de Moivre–Laplace central limit theorem). Suppose \( X_n = \text{binomial}(n, p), \) where \( p \in (0, 1) \) is fixed, and define \( Y_n \) to be its standardization. That is, \( Y_n = (X_n - \text{EX}_n)/\sqrt{\text{Var}X_n}. \) Alternatively,

\[
Y_n = \frac{X_n - np}{\sqrt{npq}}.
\]

We know that for all real numbers \( t, \)

\[
M_{X_n}(t) = (q + pe^{-t})^n.
\]

We can use this to compute \( M_{Y_n} \) as follows:

\[
M_{Y_n}(t) = \mathbb{E} \left[ \exp \left( t \cdot \frac{X_n - np}{\sqrt{npq}} \right) \right].
\]

Recall that \( X_n = I_1 + \cdots + I_n, \) where \( I_j \) is one if the \( j \)th trial succeeds; else, \( I_j = 0. \) Then, \( I_1, \ldots, I_n \) are independent binomial(1, \( p \))’s, and \( X_n - np = \sum_{j=1}^{n} (I_j - p). \) Therefore,

\[
\mathbb{E} \left[ \exp \left( t \cdot \frac{X_n - np}{\sqrt{npq}} \right) \right] = \mathbb{E} \left[ \frac{t}{\sqrt{npq}} \sum_{j=1}^{n} (I_j - p) \right]
\]

\[
= \left( \mathbb{E} \left[ \exp \left( \frac{t}{\sqrt{npq}} (I_1 - p) \right) \right] \right)^n
\]

\[
= \left( p \exp \left\{ \frac{t}{\sqrt{pq}} (1 - p) \right\} + q \exp \left\{ -\frac{t}{\sqrt{pq}} (0 - p) \right\} \right)^n
\]

\[
= \left( p \exp \left\{ t \sqrt{\frac{q}{np}} \right\} + q \exp \left\{ -t \sqrt{\frac{p}{nq}} \right\} \right)^n.
\]
According to the Taylor–MacLaurin expansion,
\[
\exp \left\{ t \sqrt{\frac{q}{np}} \right\} = 1 + t \sqrt{\frac{q}{np}} + \frac{t^2 q}{2np} + \text{smaller terms},
\]
\[
\exp \left\{ -t \sqrt{\frac{p}{nq}} \right\} = 1 - t \sqrt{\frac{p}{nq}} + \frac{t^2 p}{2nq} + \text{smaller terms}.
\]
Therefore,
\[
p \exp \left\{ t \sqrt{\frac{q}{np}} \right\} + q \exp \left\{ -t \sqrt{\frac{p}{nq}} \right\} = p \left( 1 + t \sqrt{\frac{q}{np}} + \frac{t^2 q}{2np} + \cdots \right) + q \left( 1 - t \sqrt{\frac{p}{nq}} + \frac{t^2 p}{2nq} + \cdots \right)
\]
\[
= p + t \sqrt{\frac{pq}{n} + \frac{t^2 q}{2np}} + \cdots + q - t \sqrt{\frac{pq}{n} + \frac{t^2 p}{2nq}} + \cdots
\]
\[
= 1 + \frac{t^2}{2n} + \text{smaller terms}.
\]
Consequently,
\[
M_{X_n}(t) = \left( 1 + \frac{t^2}{2n} + \text{smaller terms} \right)^n \rightarrow \exp \left( -\frac{t^2}{2} \right).
\]
We recognize the right-hand side as \( M_Y(t) \), where \( Y = N(0, 1) \). Because \( F_Y \) is continuous, this prove the central limit theorem of de Moivre: For all real numbers \( a \),
\[
\lim_{n \to \infty} P \{ Y_n \leq a \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} \, dx.
\]

2. Jointly distributed continuous random variables

**Definition 27.5.** We say that \((X, Y)\) is jointly distributed with joint density function \( f \) if \( f \) is piecewise continuous, and for all “nice” two-dimensional sets \( A \),
\[
P\{(X, Y) \in A\} = \int\int_A f(x, y) \, dx \, dy.
\]
If \((X, Y)\) has a joint density function \( f \), then:

1. \( f(x, y) \geq 0 \) for all \( x \) and \( y \);
2. \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1 \).

Any function \( f \) of two variables that satisfies these properties will do.
Example 27.6 (Uniform joint density). Suppose $A$ is a subset of the plane that has a well-defined finite area $|A| > 0$. Define

$$f(x, y) = \begin{cases} \frac{1}{|A|} & \text{if } (x, y) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f$ is a joint density function, and the corresponding random vector $(X, Y)$ is said to be distributed uniformly on $A$. Moreover, for all planar sets $E$ with well-defined areas,

$$P\{(X, Y) \in E\} = \int_E \int_{E \cap A} \frac{1}{|A|} \, dx \, dy = \frac{|E \cap A|}{|A|}.$$ 

See Figure 1.

Example 27.7. Suppose $(X, Y)$ has joint density

$$f(x, y) = \begin{cases} Cxy & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$
2. Jointly distributed continuous random variables

Let us first find $C$, and then $P\{X \leq 2Y\}$. To find $C$:

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{x} Cxy \, dy \, dx$$

$$= C \int_{0}^{1} x \left( \int_{0}^{x} y \, dy \right) \, dx = C \int_{0}^{1} \frac{x^{3}}{2} \, dx = C \frac{1}{2} \cdot \frac{1}{3} = C \frac{1}{6}$$

Therefore, $C = 8$, and hence

$$f(x, y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$P\{X \leq 2Y\} = P\{(X, Y) \in A\} = \iiint_{A} f(x, y) \, dx \, dy,$$

where $A$ denotes the collection of all points $(a, b)$ in the plane such that $a \leq 2b$. Therefore,

$$P\{X \leq 2Y\} = \int_{0}^{1} \int_{x/2}^{x} 8xy \, dy \, dx = \frac{3}{32}.$$ 

See Figure 2.