Math 5010–1, Spring 2007
Final Examination Practice

1. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed.

   (a) Prove that if the X_j's are $N(0\,,1)$ then $X_1+\cdots+X_n$ is $N(0\,,n)$. Hint: $M_{N(0,\sigma^2)}(t) = \exp(-\sigma^2 t^2/2)$.

   (b) Prove that if the X_j's are Poisson(λ), then $X_1 + \cdots + X_n$ is Poisson($n\lambda$). Hint: $M_{\text{Poisson}(\lambda)}(t) = \exp\{\lambda(e^t - 1)\}$.

2. Let $X_1, X_2, \ldots, X_{20}$ be independent Poisson random variables with mean one.

   (a) Compute $\text{Var}(X_1)$.

   (b) Use the preceding and the central limit theorem to approximate $P\{\sum_{i=1}^{20} X_i > 15\}$.

3. A point $(X\,,Y)$ is chosen at random according to the following (joint) probability mass function:

X \ **Y**	0	1	2
0	1/12	1/12	1/12
1	1/12	1/12	1/12
2	0	1/8	1/8
3	0	0	1/4

   (a) Compute the mass function of $W = XY$.

   (b) Compute $P\{X > Y\}$.

4. Suppose X and Y are independent, both distributed uniformly on $[0\,,1]$. Then prove that

$$E\left(|X - Y|^{\alpha}\right) = \frac{2}{(\alpha+1)(\alpha+2)} \quad \text{for } \alpha > 0.$$

5. A fair die is cast ten times, and the total number of rolled dots is summed up; call this sum X. Find $E[X]$.

6. If X and Y are independent and identically distributed with $\mu = E[X]$ and $\sigma^2 = \text{Var}(X)$, then compute $E[(X - Y)^2]$.

7. (a) Compute the moment generating function of a random variable X whose mass function is as follows:

$$p(0) = \frac{1}{3}, \quad p(2) = \frac{1}{6}, \quad p(-1) = \frac{1}{2}.$$

   (b) Suppose Y is a random variable whose moment generating function is

$$M_{Y}(t) = \frac{1}{2}e^t + \frac{1}{6}e^{-2t} + \frac{1}{3}e^{5t}.$$

   Compute the mass function of Y.

8. The density function of X is given by

$$f(x) = a + bx^2 \qquad 0 \le x \le 1.$$

If $E[X] = 3/5$ then find a and b.

9. Let $X_1, X_2, \ldots, X_{20}$ be independent random variable, all uniformly distributed on $(0\,,1)$.

   (a) Compute the mean μ and variance σ^2 of the X_j's.

   (b) Use the central limit theorem, and a normal table, to approximate the probability that $X_1 + \cdots + X_{20}$ remains between 8.5 and 11.7.

10. You have 4 light bulbs whose lifetimes are independent normal random variables with mean 100 (hours) and standard deviation 5 (hours). Use the central limit theorem to approximate the probability that your 4 light bulbs together live for at least 402 hours. Your answer should be written as an explicit integral involving the normal density; not a numerical answer.

11. Consider a random vector $(X\,,Y)$. We know that X is exponentially distributed with parameter 1; i.e.,

$$f_X(x) = \begin{cases} e^{-x}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x > 0$ be any fixed positive number, and suppose that conditionally on the event $\{X = x\}$, Y is exponentially distributed with parameter $1/x$. That is,

$$f_{Y|X}(y\,|\,x) = \begin{cases} \frac{1}{x}e^{-y/x}, & y \ge 0, \\ 0, & \qquad\qquad \text{otherwise,} \end{cases}$$

   (a) Compute the density function of $(X\,,,Y)$.

   (b) Compute the expected value of Y.

   (c) What is $P\{Y > X\}$?

   (d) Find the density function of $(X^2\,,Y^2)$.

12. Suppose X_1, X_2, X_3 and X_4 are independent with common mean 1 and common variance 2. Compute $\mathrm{Cov}(X_1 + X_2\,, X_2 + X_3)$.

13. Suppose X is a standard normal random variable. What is the density of $1/X^2$?

14. Let X denote a random variable with the following probability mass function.

$$p(j) = 2^{-j}, \qquad j = 1, 2, \ldots.$$

   (a) Compute the moment generating function of X. (Hint: You may use the fact that $\sum_{j=0}^{\infty} r^j = 1/(1-r)$ for $0 < r < 1$.)

   (b) Use your answer to part (a) to compute the expectation of X.

15. A fair die is cast 112 times. What is the probability that we roll two dots at least twice?

16. Suppose (X, Y) is distributed uniformly on the square with corners at $(0\,,0)$, $(1\,,0)$, $(0\,,1)$, and $(1\,,1)$. Compute $\mathrm{P}\{Y > aX\}$ for all real numbers a.

17. A point $(X\,,Y)$ is picked at random, uniformly from the square whose corners are at $(0\,,0)$, $(1\,,0)$, $(0\,,1)$ and $(1\,,1)$.

   (a) Compute $\mathrm{Cov}(X\,,Y)$.

   (b) Compute $P\{Y > \frac{1}{2} + X\}$.

18. A pair of fair dice are cast independently from one another. If you are told that the total sum of the dots is 10, then find the probability that the first die was 6.