Lecture 5

1. Independence

- Events A and B are said to be *independent* if

$$P(A \cap B) = P(A)P(B).$$

Divide both sides by $P(B)$, if it is positive, to find that A and B are independent if and only if

$$P(A \mid B) = P(A).$$

"Knowledge of B tells us nothing new about A."

Two experiments are *independent* if A_1 and A_2 are independent for all outcomes A_j of experiment j.

Example 5.1. Toss two fair coins; all possible outcomes are equally likely. Let H_j denote the event that the jth coin landed on heads, and $T_j = H_j^c$. Then,

$$P(H_1 \cap H_2) = \frac{1}{4} = P(H_1)P(H_2).$$

In fact, the two coins are independent because $P(T_1 \cap T_2) = P(T_1 \cap H_2) = P(H_1 \cap H_2) = 1/4$ also. Conversely, if two fair coins are tossed independently, then all possible outcomes are equally likely to occur. What if the coins are not fair, say $P(H_1) = P(H_2) = 1/4$?

- Three events A_1, A_2, A_3 are *independent* if any two of them. Events A_1, A_2, A_3, A_4 are independent if any three of are. And in general, once we have defined the independence of $n-1$ events, we define n events $A_1, \ldots, A_n$ to be *independent* if any $n-1$ of them are independent.
- One says that n experiments are *independent*, for all $n \geq 2$, if any $n-1$ of them are independent.

You should check that this last one is a well-defined (albeit inductive) definition.

2. Gambler's ruin formula

You, the "Gambler," are playing independent repetitions of a fair game against the "House." When you win, you gain a dollar; when you lose, you lose a dollar. You start with k dollars, and the House starts with K dollars. What is the probability that the House is ruined before you?

Define P_j to be the conditional probability that when the game ends you have $K + j$ dollars, given that you start with j dollars initially. We want to find P_k.

Two easy cases are: $P_0 = 0$ and $P_{k+K} = 1$.

By Theorem 4.4 and independence,

$$P_j = \frac{1}{2}P_{j+1} + \frac{1}{2}P_{j-1} \qquad \text{for } 0 < j < k + K.$$

In order to solve this, write $P_j = \frac{1}{2}P_j + \frac{1}{2}P_j$, so that

$$\frac{1}{2}P_j + \frac{1}{2}P_j = \frac{1}{2}P_{j+1} + \frac{1}{2}P_{j-1} \qquad \text{for } 0 < j < k + K.$$

Multiply both side by two and solve:

$$P_{j+1} - P_j = P_j - P_{j-1} \qquad \text{for } 0 < j < k + K.$$

In other words,

$$P_{j+1} - P_j = P_1 \qquad \text{for } 0 < j < k + K.$$

This is the simplest of all possible "difference equations." In order to solve it you note that, since $P_0 = 0$,

$$\begin{aligned} P_{j+1} &= (P_{j+1} - P_j) + (P_j - P_{j-1}) + \cdots + (P_1 - P_0) \qquad \text{for } 0 < j < k + K \\ &= (j + 1)P_1 \qquad \text{for } 0 < j < k + K. \end{aligned}$$

Apply this with $j = k + K - 1$ to find that

$$1 = P_{k+K} = (k + K)P_1, \quad \text{and hence} \quad P_1 = \frac{1}{k + K}.$$

Therefore,

$$P_{j+1} = \frac{j + 1}{k + K} \qquad \text{for } 0 < j < k + K.$$

Set $j = k - 1$ to find the following:

Theorem 5.2 (Gambler's ruin formula)**.** *If you start with k dollars, then the probability that you end with $k+K$ dollars before losing all of your initial fortune is $k/(k + K)$ for all $1 \le k \le K$.*

3. Conditional probabilities as probabilities

Suppose B is an event of positive probability. Consider the conditional probability distribution, $\mathrm{Q}(\cdots) = \mathrm{P}(\cdots \mid B)$.

Theorem 5.3. Q *is a probability on the new sample space B. [It is also a probability on the larger sample space Ω, why?]*

Proof. Rule 1 is easy to verify: For all events A,

$$0 \le \mathrm{Q}(A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \le \frac{P(B)}{\mathrm{P}(B)} = 1,$$

because $A \cap B \subseteq B$ and hence $\mathrm{P}(A \cap B) \le \mathrm{P}(B)$.

For Rule 2 we check that

$$\mathrm{Q}(B) = \mathrm{P}(B \mid B) = \frac{\mathrm{P}(B \cap B)}{\mathrm{P}(B)} = 1.$$

Next suppose $A_1, A_2, \ldots$ are disjoint events. Then,

$$\mathrm{Q}\left(\bigcup_{n=1}^{\infty} A_n\right) = \frac{1}{\mathrm{P}(B)}\,\mathrm{P}\left(\bigcup_{n=1}^{\infty} A_n \cap B\right).$$

Note that $\cup_{n=1}^{\infty} A_n \cap B = \cup_{n=1}^{\infty}(A_n \cap B)$, and $(A_1 \cap B), (A_2 \cap B), \ldots$ are disjoint events. Therefore,

$$\mathrm{Q}\left(\bigcup_{n=1}^{\infty} A_n\right) = \frac{1}{\mathrm{P}(B)} \sum_{N=1}^{\infty} \mathrm{P}\left(A_n \cap B\right) = \sum_{n=1}^{\infty} \mathrm{Q}(A_n).$$

This verifies Rule 4, and hence Rule 3. □