

1. Easy cards

There are 52 cards in a deck. You deal two cards, all pairs equally likely.

Math model: Ω is the collection of all pairs [drawn without replacement from an ordinary deck]. What is $|\Omega|$? To answer this note that $2|\Omega|$ is the number of all possible ways to give a pair out; i.e., $2|\Omega| = 52 \times 51$, by the principle of counting. Therefore,

$$|\Omega| = \frac{52 \times 51}{2} = 1326.$$

- The probability that the second card is an ace is $(4 \times 51)/2 = 102$ divided by 1326. This probability is $\simeq 0.0769$
- The probability that both cards are aces is $(4 \times 3)/2 = 6$ divided by 1326, which is $\simeq 0.0045$.
- The probability that both cards are the same is $P\{\text{ace and ace}\} + \dots + P\{\text{King and King}\} = 13 \times 0.0769 \simeq 0.0588$.

2. The birthday problem

n people in a room; all birthdays are equally likely, and assigned at random. What are the chances that no two people in the room are born on the same day? You may assume that there are 365 days a years, and that there are no leap years.

Let $p(n)$ denote the probability in question.

To understand this consider finding $p(2)$ first. There are two people in the room.

The sample space is the collection of all pairs of the form (D_1, D_2) , where D_1 and D_2 are birthdays. Note that $|\Omega| = 365^2$ [principle of counting].

In general, Ω is the collection of all “ n -tuples” of the form (D_1, \dots, D_n) where the D_i ’s are birthdays; $|\Omega| = 365^n$. Let A denote the collection of all elements (D_1, \dots, D_n) of Ω such that all the D_i ’s are distinct. We need to find $|A|$.

To understand what is going on, we start with $n = 2$. In order to list all the elements of A , we observe that we have to assign two separate birthdays. [Forks = first birthday; knives = second birthday]. There are therefore 365×364 outcomes in A when $n = 2$. Similarly, when $n = 3$, there are $365 \times 364 \times 363$, and in general, $|A| = 365 \times \dots \times (365 - n + 1)$. **Check this with induction!**

Thus,

$$p(n) = \frac{|A|}{|\Omega|} = \frac{365 \times \dots \times (365 - n + 1)}{365^n}.$$

For example, check that $p(10) \simeq 0.88$, which may seem very high at first.

3. An urn problem

n purple and n orange balls are in an urn. You select balls at random [without replacement]. What are the chances that they have different colors?

Here, Ω denotes the collection of all pairs of colors. Note that $|\Omega| = 2n(2n - 1)$ [principle of counting].

$$P\{\text{two different colors}\} = 1 - P\{\text{the same color}\}.$$

Also,

$$P\{\text{the same color}\} = P(P_1 \cap P_2) + P(O_1 \cap O_2),$$

where O_j denotes the event that the j th ball is orange, and P_k the event that the k th ball is purple. The number of elements of $P_1 \cap P_2$ is $n(n - 1)$; the same holds for $O_1 \cap O_2$. Therefore,

$$\begin{aligned} P\{\text{different colors}\} &= 1 - \left[\frac{n(n - 1)}{2n(2n - 1)} + \frac{n(n - 1)}{2n(2n - 1)} \right] \\ &= \frac{n}{2n - 1}. \end{aligned}$$

In particular, regardless of the value of n , we always have

$$P\{\text{different colors}\} > \frac{1}{2}.$$